BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

EARLE R. HEDRICK

DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN

THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER

JOHN R. KLINE HENRY L. RIETZ

CAROLINE E. SEELY

VOLUME XXXVII, NUMBER 10

OCTOBER, 1931

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1931

PUBLISHED MONTHLY

Whole No. 385 $9.00 a Year

Entered as second class matter April 24, 1926 at the post office at Menasha, Wis., under the act of August 24, 1912.

PUBLICATIONS OF THIS SOCIETY

I. THE TRANSACTIONS. To members, $6.75; to others, $9.00.

Volumes 1-14 and 16-21: to members, $5.00; to others, $7.00.

Volumes 15, and 22-32: to members, $6.00; to others, $8.00.

II. THE BULLETIN. To non-members, $9.00.
Members of the Society receive the Bulletin if they have paid their dues, $5.00 of which is for subscription to the Bulletin.

Volumes 1-11 (and the three volumes of the Bulletin of the New York Mathematical Society), are out of print. The Society is glad to receive odd volumes or even odd numbers of these volumes.

Volumes 12-36: to members, $5.25; to others, $7.00.

III. THE AMERICAN JOURNAL OF MATHEMATICS
Published by this Society and the Johns Hopkins University. Send orders to The Johns Hopkins Press.

To members, $5.75; to others, $7.50.

Complete sets (50 volumes): $350.00.

IV. THE COLLOQUIUM PUBLICATIONS.
Published in December, 1930:

S. Lefschetz, *Topology*, $4.50. (Volume XII of the Colloquium Series.)

Published in April, 1931:

O. Veblen, *Analysis Situs*. Second edition. $2.00. (Part II of Volume V of the Colloquium Series.)

For earlier issues see a fly-leaf of the September issue.

Orders may be addressed to
450-458 AHNAIP ST., MENASHA, WIS., OR
501 WEST 116TH STREET, NEW YORK, N.Y.

On sale also by the following official agents of this Society:

Bowes & Bowes, 1 Trinity St., Cambridge, England.

Hirschwaldsche Buchhandlung, Unter den Linden, 68, Berlin.

Librairie Scientifique Albert Blanchard, 3 bis, Place de la Sorbonne, Paris, V, France.

Libreria Editrice Nicola Zanichelli, Bologna, Italy.

ANALYTIC FUNCTIONS AND MATHEMATICAL PHYSICS*

BY G. Y. RAINICH

1. *Some Properties of Analytic Functions.* We begin by reviewing briefly some fundamental points in the theory of analytic functions in a form which will be convenient for further references. Departing slightly from customary notations, we shall write $w=v+iu$, and we shall consider the theory primarily as the theory of a system of differential equations

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

which are called the Cauchy-Riemann equations.

We shall not enter into various fine points which arise in the discussion, but we may mention that the functions u and v, as well as the other functions which appear later, must be assumed to be differentiable, that is, to possess complete differentials in the sense of Stolz.

(a) One point of view often taken in applications is that we have a vector or, rather a vector field, of components (u, v), and that the differential equations express the fact that the rotation (curl) and the divergence of this vector are zero.

Another point of view, which we shall find extremely useful, is that we have *two* vectors f and r, whose components are $f_1=u$, $f_2=v$, and $r_1=v$, $r_2=-u$, respectively, and that the differential equations express the fact that the divergences of both are zero. These two vectors, as the relations

$$f_1=-r_2, \qquad f_2=r_1, \tag{2}$$

show, are perpendicular and of equal length, so that we may say that the theory of analytic functions is the theory of two equal and perpendicular vectors in the plane, with zero divergences.

(b) The differential equations may be considered as integrability conditions. They are equivalent to the vanishing of certain

* This paper covers in part the subject matter of a symposium lecture given by the author before the Society in Chicago, April 3, 1931.

contour integrals, taken around contours inside of which the functions are regular.

(c) If the functions are not regular inside a contour but have singularities in one point, the integrals do not in general vanish, but furnish two real numbers (which taken together as a complex number form a residue) and which characterize to a certain extent the singularity. The simplest type of singularity is shown by the function $w=ci/z$, for which in our notations we have

$$u = \frac{cx}{r^2}, \qquad v = \frac{cy}{r^2}; \tag{3}$$

the residue in this case is equal to ci. (Here c is a real number and $r^2=x^2+y^2$.)

(d) By elimination of one of the functions (u, v), we obtain for the other the second-order (Laplace) equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{4}$$

(e) There are certain second-degree quantities that might be discussed in connection with an analytic function; these are

$$W = \tfrac{1}{2}w^2 = \tfrac{1}{2}(v^2 - u^2) + iuv, \tag{5}$$

half the square of the function; and

$$Q = w \cdot \bar{w} = u^2 + v^2 = \tfrac{1}{2}(f_1^2 + f_2^2 + r_1^2 + r_2^2), \tag{6}$$

the square of the modulus, or the norm of the function. By the fact that (u, v) satisfy the differential equations (1) certain conditions are imposed on these second-degree quantities, and we shall now write them down.

For W it is easy; the statement that w is an analytic function means the same as the statement that $\frac{1}{2}w^2$ is an analytic function, and the latter statement is equivalent to the relations

$$\begin{cases} \dfrac{\partial \frac{1}{2}(u^2 - v^2)}{\partial x} + \dfrac{\partial uv}{\partial y} = 0, \\ \dfrac{\partial uv}{\partial x} + \dfrac{\partial \frac{1}{2}(v^2 - u^2)}{\partial y} = 0. \end{cases} \tag{7}$$

This system of differential equations is then equivalent to system (1), and it is easy to verify this fact directly.

The situation is more complicated with respect to Q. The easiest way to arrive at the differential equation which must be satisfied by Q as a result of the fact that u, v satisfy (1) seems to be the consideration of $\log w^2$. This must be analytic if w is, and therefore the real part of $\log w^2$ must satisfy the Laplace equation. But the real part of $\log w^2$ is $\log w\bar{w}$, so that we have

$$\frac{\partial^2 \log Q}{\partial x^2} + \frac{\partial^2 \log Q}{\partial y^2} = 0$$

or

$$Q \cdot \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2}\right) = \left(\frac{\partial Q}{\partial x}\right)^2 + \left(\frac{\partial Q}{\partial y}\right)^2. \tag{8}$$

It is interesting to note that this equation is non-linear, and that it therefore gives an example of a non-linear equation which is a consequence of a system of linear equations. We next ask ourselves whether the condition expressed by the last equation is sufficient, in other words whether every function Q satisfying this condition may be considered as the norm of an analytic function.

In the first place it is clear that a function is not entirely determined by its norm. If the function $w=v+iu$ has the norm Q, any function $V+iU$, for which

$$U = u\cos\phi - v\sin\phi, \quad V = u\sin\phi + v\cos\phi, \tag{9}$$

where ϕ is an arbitrary function of x and y, will have the same norm. The question then reduces to this: are there among the functions $V+iU$ some that are analytic? An easy calculation leads to the result that the equations

$$\begin{aligned} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + u\cdot\phi_1 + v\cdot\phi_2 &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - v\cdot\phi_1 + u\cdot\phi_2 &= 0, \end{aligned} \tag{10}$$

with $\phi_1=\partial\phi/\partial x$, $\phi_2=\partial\phi/\partial y$, must be satisfied. Further calculation shows that the integrability condition for these equations is exactly the above equation (7) for Q.

(f) An analytic function can be developed into a power series. In view of an analogy that we want to establish later the following way to introduce this series might find its place here. We first write the Cauchy-Riemann equations (1) in the form $\partial w/\partial x + i\partial w/\partial y = 0$, and, passing to polar coordinates, we have

$$r\frac{\partial w}{\partial r} + i\frac{\partial w}{\partial \theta} = 0. \tag{11}$$

Next, making use of the fact that the equation does not contain θ explicitly, and therefore must allow solutions of the form

$$Pe^{ik\theta}, \tag{12}$$

where P is independent of θ, we find for P the equation

$$r\frac{\partial P}{\partial r} - kP = 0,$$

which has a solution $P = r^k$, so that

$$w = r^k e^{ik\theta} = z^k \tag{13}$$

is a solution of the original equation. If we require the solution to be one-valued, k must be an *integer*, and if we want it to be continuous (at the origin), k must be *non-negative*. A linear combination of a finite number of such solutions is a polynomial, and the general solution may be presented as a linear combination of an infinite number of such solutions which we may consider as the limit of a (uniformly convergent) sequence of polynomials, or, if you prefer, as a power series.

2. *The Volterra Theory.* We pass now to generalizations. We saw that the theory of analytic functions may be considered as the theory of two equal and perpendicular vectors with vanishing divergences. We may try to extend this to the three-dimensional space. If we take two vectors f and r and write down the conditions for their equality and perpendicularity, we find

$$f_1^2 + f_2^2 + f_3^2 = r_1^2 + r_2^2 + r_3^2, \quad f_1 \cdot r_1 + f_2 \cdot r_2 + f_3 \cdot r_3 = 0 \tag{14}$$

and although the differential equations

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0, \quad \frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial y} + \frac{\partial r_3}{\partial z} = 0 \tag{15}$$

are linear, the system as a whole is *not*. The same is true in the two-dimensional case, of course, but there the equations are reduced to linear equations easily. We shall not consider now the question whether these equations can be reduced to linear equations (see §6); trying to keep more or less to the historical order we shall outline a way out from this difficulty, which leads to Volterra's theory of conjugate functions dated back to 1889.*

We may consider a vector as a finite portion of a directed line (or curve), and its components as the lengths of the projections of this finite portion on the coordinate axes. If now we consider a finite portion of a plane, or surface, as a *surface vector*, and as its components the *areas* of its projections on the coordinate planes, we have a new object to operate upon. The general case may be reduced to that of a triangle with vertices at $(0, 0, 0)$, (x_1, x_2, x_3), (y_1, y_2, y_3). The areas of the projections are the determinants of the matrix

$$\frac{1}{2}\begin{Vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{Vmatrix},$$

which we may denote by R_{23}, R_{31}, R_{12} or L, M, N, or in general

$$R_{ij} = \frac{1}{2}\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}.$$

It may seem sufficient to consider only the above three components; however, it is more convenient not to restrict i and j, but to use all nine combinations, introducing the relation $X_{ij}+X_{ji}=0$. These nine may be arranged into a square matrix

$$(16) \qquad \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}, \quad \begin{pmatrix} O & N & -M \\ -N & O & L \\ M & -L & O \end{pmatrix},$$

so that a surface vector is represented by a matrix possessing the property of antisymmetry.

Consider now a *line* vector $X=F_1$, $Y=F_2$, $Z=F_3$ with the above *surface* vector. Conditions of perpendicularity and nu-

* V. Volterra, *Sulle funzioni conjugate*, Rendiconti dei Lincei, (4), vol. 5 (1889), pp. 599–611. See also Rendiconti di Palermo, vol. 3 (1889), pp. 260–272.

merical equality may be shown to result simply in the equations

$$R_{23} = F_1, \quad R_{31} = F_2, \quad R_{12} = F_3, \tag{17}$$

which should be compared with the equations (2). If now we impose the vanishing of divergences

$$\sum \frac{\partial F_\alpha}{\partial x_\alpha} = 0, \quad \sum \frac{\partial R_{i\alpha}}{\partial x_\alpha} = 0,$$

and write out everything without indices in terms of X, Y, Z, x, y, z, we obtain a system of four equations

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0, \quad \frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}, \quad \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}. \tag{18}$$

This is an analog for three-space of the Cauchy-Riemann system. It is worth noting that setting $Z=0$ we get two functions X and Y depending on x and y alone which satisfy the equations

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0, \qquad \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y},$$

that is, exactly the Cauchy-Riemann equations for $X=u$, $Y=v$. We further note that the elimination of two of the three functions X, Y, Z from the above four equations leads as in (d), §1, to the Laplace equation in three dimensions. We have thus a theory in three-space which may be considered as a generalization of the theory of analytic functions, and which is essentially the theory of two equal and perpendicular vectors with vanishing divergences, one of these vectors being a line vector, and the other a surface vector.

There are no essential difficulties (except that of losing the help of intuition) in extending the theory to any number of dimensions. We shall only take up the case $n=4$. Here we may consider the case of two perpendicular *surface* vectors with vanishing divergences. There are here *six* coordinate planes, and therefore every surface vector has six components. They are known as *six-vectors*. We may write, using notations similar to those used above, f_{ij} for the components, with the condition $f_{ij}+f_{ji}=0$. In four-space, this will mean a square four-rowed matrix, whose sixteen elements reduce to six as a result of these relations. The conditions of numerical equality and perpendicu-

larity for two such vectors f and r again reduce to linear relations, namely,

(19) $f_{41} = r_{23}$, $f_{42} = r_{31}$, $f_{43} = r_{12}$, $f_{23} = r_{41}$, $f_{31} = r_{42}$, $f_{12} = r_{43}$,

which should be compared with (2) and (18). The divergence equations together with these relations constitute a linear system which is analogous to, and a generalization of, the Cauchy-Riemann system (1). This system may be written as

$$(20) \qquad \sum \frac{\partial f_{i\alpha}}{\partial x_\alpha} = 0, \quad \sum \frac{\partial r_{i\alpha}}{\partial x_\alpha} = 0,$$

or, in full, if we set

(21) $f_{14} = X$, $f_{24} = Y$, $f_{34} = Z$, $f_{23} = L$, $f_{31} = M$, $f_{12} = N$,

$$(22) \quad \begin{cases} \dfrac{\partial Z}{\partial y} - \dfrac{\partial Y}{\partial z} + \dfrac{\partial L}{\partial t} = 0, & \dfrac{\partial N}{\partial y} - \dfrac{\partial M}{\partial z} + \dfrac{\partial X}{\partial t} = 0, \\ \dfrac{\partial X}{\partial z} - \dfrac{\partial Z}{\partial x} + \dfrac{\partial M}{\partial t} = 0, & \dfrac{\partial L}{\partial z} - \dfrac{\partial N}{\partial x} + \dfrac{\partial Y}{\partial t} = 0, \\ \dfrac{\partial Y}{\partial x} - \dfrac{\partial X}{\partial y} + \dfrac{\partial N}{\partial t} = 0, & \dfrac{\partial M}{\partial x} - \dfrac{\partial L}{\partial y} + \dfrac{\partial Z}{\partial t} = 0, \\ \dfrac{\partial L}{\partial x} + \dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial z} = 0, & \dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z} = 0. \end{cases}$$

Here again the elimination of all but one component leads to a Laplace equation.

As I have said, analogous considerations apply to spaces of any number of dimensions, and when the sum of the numbers of dimensions of the two vectors is equal to the number of dimensions of the space the system reduces to a linear system—this is the case studied by Volterra.* Volterra's work has been followed up by Lagally, de Donder, Dixon,† and a few others, but very much remains to be done. Many features of the theory of analytic functions are preserved, but not all. Complex numbers or

* Volterra, loc. cit.

† M. Lagally, Münchener Berichte, 1917. Th. de Donder, Bulletin des Sciences de Belgique, 1906, pp. 400–409. A. C. Dixon, Quarterly Journal, vol. 35 (1904), pp. 283–296.

hypercomplex numbers are not used except in the four-dimensional case by Dixon. The three-dimensional theory bears the same relationship to the Newtonian potential as the theory of analytic functions to the logarithmic potential, and some results of potential theory can be directly translated into this theory, but there are some questions which have to be treated independently. Of course, the differential equations may be replaced by integral conditions, by the vanishing of certain integrals taken over surfaces surrounding volumes in which the functions are regular. An extension of the Cauchy integral formula can be proved for all cases. There are different types of singularities, point singularities, line singularities, etc. The theory of residues presents a particular fascination. Expansions analogous to power series exist; in the three-dimensional case they are essentially developments into series of harmonic functions. But we shall abandon now the purely mathematical developments and pass to applications.

3. *Analogous Situations in Physics.* We shall not go farther back than Newton in tracing the development in mathematical physics. With the name of Newton we associate a great many things, but here we are interested in particular in two. First, in the general laws of mechanics stating that the time rate of change of the momentum is equal to force

$$\frac{dmu}{dt} = X, \quad \frac{dmv}{dt} = Y, \quad \frac{dmw}{dt} = Z; \tag{23}$$

and, second, in the special case (inverse square law) in which the force components are given by

$$X = \frac{cx}{r^3}, \quad Y = \frac{cy}{r^3}, \quad Z = \frac{cz}{r^3}, \tag{24}$$

which should be compared with (3). We have here the attracting point, and the attracted point, both as discrete points. One of the trends of physics has been the transition from the consideration of discrete points to the consideration of fields, that is, continuously distributed quantities. We shall follow this trend separately for the right-hand sides and the left-hand sides of the equations, that is, the forces and matter.

First, as to the forces, we find easily that the force components given by the explicit formulas (24) satisfy the differential equations

$$(25)\quad \frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0,\quad \frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x},\ \frac{\partial X}{\partial z}=\frac{\partial Z}{\partial x},\ \frac{\partial Y}{\partial z}=\frac{\partial Z}{\partial y}.$$

This is at the same time a generalization of the inverse square law and its source. We can get the inverse square law back from these differential equations by solving them under the additional condition of symmetry around a point. In passing from an explicit expression for the law of force to the partial differential equations we shifted our attention, without changing any essential features of the situation, from the singularities of the field to the field connected with these singularities.

We may do a similar thing to the left-hand sides of our equations, and introduce a continuous fluid instead of a discrete particle (the case of a particle may then be obtained as a limiting case of infinite density). Mathematically, this transition is a transition from ordinary to partial differential equations. In the simplest case, that of steady motion, when density is constant, and there exists a potential of velocities, the equations satisfied by the components of velocity are

$$(26)\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x},\ \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x},\ \frac{\partial v}{\partial z}=\frac{\partial w}{\partial y}.$$

These equations were known in the 18th century; they appear for instance, in a memoir of Lagrange* in 1760; the remarkable thing is that these equations of the simplest motion of a fluid are the same as those of the simplest field of force. Furthermore these equations are exactly the case $n=3$ of the Volterra theory (§2) and thus a generalization of the equations (1) of the theory of analytic functions. The inverse square law appears as the simplest singularity corresponding to $w=c/z$ (compare formulas (3) and (24)); the exponent in the denominator is equal to the number of dimensions in each case. The residue corresponds to the mass, or electric charge of the attracting point.

In considering cases more general than the simple one just mentioned the ways of the forces and those of the velocities

* Lagrange, Oeuvres, vol. 1, p. 442.

part, at least for a time. For hydrodynamics we have to use in the general case the equations of Euler, which we take in the following form:

$$(27)\qquad \begin{cases} \dfrac{\partial \rho u^2}{\partial x}+\dfrac{\partial \rho uv}{\partial y}+\dfrac{\partial \rho uw}{\partial z}+\dfrac{\partial \rho u}{\partial t}=X+\dfrac{\partial p}{\partial x}, \\ \dfrac{\partial \rho vu}{\partial x}+\dfrac{\partial \rho v^2}{\partial y}+\dfrac{\partial \rho vw}{\partial z}+\dfrac{\partial \rho v}{\partial t}=Y+\dfrac{\partial p}{\partial y}, \\ \dfrac{\partial \rho wu}{\partial x}+\dfrac{\partial \rho wv}{\partial y}+\dfrac{\partial \rho w^2}{\partial z}+\dfrac{\partial \rho w}{\partial t}=Z+\dfrac{\partial p}{\partial z}, \\ \dfrac{\partial \rho u}{\partial x}+\dfrac{\partial \rho v}{\partial y}+\dfrac{\partial \rho w}{\partial z}+\dfrac{\partial \rho}{\partial t}=0, \end{cases}$$

where p is the pressure and ρ is the density. The first three of these equations correspond to the three equations (23), and are nothing but their translation into the language of continuous distribution; the last one is a translation of the equation $dm/dt=0$, which is usually not written out explicitly in the discrete case.

The first thing we notice about these equations is that they deal not with the quantities u, v, and w, but with quadratic expressions in them. In the simplest case $n=2$, replacing the pressure p by $\frac{1}{2}(u^2+v^2)$ (internal pressure), the density by 1, and the forces by zero, we get the equations

$$(28)\qquad \frac{\partial \frac{1}{2}(u^2-v^2)}{\partial x}+\frac{\partial uv}{\partial y}=0, \quad \frac{\partial uv}{\partial x}+\frac{\partial \frac{1}{2}(v^2-u^2)}{\partial y}=0,$$

which are exactly the equations (7) for what we have called W in the theory of analytic functions. With the same assumptions, but for $n=3$, we get

$$(29)\qquad \begin{cases} \dfrac{\partial \frac{1}{2}(u^2-v^2-w^2)}{\partial x}+\dfrac{\partial uv}{\partial y}+\dfrac{\partial uw}{\partial z}=0, \\ \dfrac{\partial vu}{\partial x}+\dfrac{\partial \frac{1}{2}(-u^2+v^2-w^2)}{\partial y}+\dfrac{\partial vw}{\partial z}=0, \\ \dfrac{\partial wu}{\partial x}+\dfrac{\partial wv}{\partial y}+\dfrac{\partial \frac{1}{2}(-u^2-v^2+w^2)}{\partial z}=0. \end{cases}$$

We naturally ask about the relation between these differential equations for the quadratic quantities and the differential equations written out before for the linear quantities. As in the case $n=2$, the equations for the second-degree quantities are consequences of those for the linear quantities, but in contradistinction from that case we cannot pass here from (29) to (26); this raises the question: what additional conditions have to be imposed on the quadratic quantities so that the linear equations would be satisfied?

The form of Euler's set of equations suggests the consideration of the case $n=4$, by considering time as the fourth coordinate and t as the fourth component of the vector (u, v, w, t). Taking this suggestion seriously means making the first step toward the special theory of relativity. Without entering into details we must mention that velocity is represented in this theory by a four-vector, whose components we may denote by u_i, and that if we consider from this point of view matter in the absence of forces, we arrive at a set of equations which constitute the generalization for four dimensions of (28) and (29).

We shall now return to the consideration of forces. We spoke of the inverse square law without mentioning the *physical nature* of the forces. Historically, the gravitational field was the one for which in the time of Newton the inverse square law was formulated. But we are now more interested in the electrostatic and magnetostatic fields to which the same law has been applied in the nineteenth century. To each of these fields taken separately that law, and therefore the equation (25), applies. As soon as the time is introduced, however, that is, as soon as we begin to consider fields changing with time, interaction between these fields appears, and we have to consider them together. The result may be formulated as follows. In the static case the quantities $\partial Y/\partial z-\partial Z/\partial y$, $\partial Z/\partial x-\partial X/\partial z$, $\partial X/\partial y-\partial Y/\partial x$ and $\partial M/\partial z-\partial N/\partial y$, $\partial N/\partial x-\partial L/\partial z$, $\partial L/\partial y-\partial M/\partial x$ were zero. For the case when the fields change with time, Faraday's discoveries coupled with Maxwell's imagination resulted in equating these quantities (I omit the factor of proportionality), respectively, to $\partial L/\partial t$, $\partial M/\partial t$, $\partial N/\partial t$ and $-\partial X/\partial t$, $-\partial Y/\partial t$, $-\partial Z/\partial t$, so that together with the equations expressing the vanishing of the divergences we have the following set of equations for electromagnetic forces in matter-free space (*Maxwell's equations*):

$$(30)\qquad \begin{cases} \dfrac{\partial Y}{\partial z}-\dfrac{\partial Z}{\partial y}-\dfrac{\partial L}{\partial t}=0, & \dfrac{\partial M}{\partial z}-\dfrac{\partial N}{\partial y}+\dfrac{\partial X}{\partial t}=0, \\ \dfrac{\partial Z}{\partial x}-\dfrac{\partial X}{\partial z}-\dfrac{\partial M}{\partial t}=0, & \dfrac{\partial N}{\partial x}-\dfrac{\partial L}{\partial z}+\dfrac{\partial Y}{\partial t}=0, \\ \dfrac{\partial X}{\partial y}-\dfrac{\partial Y}{\partial x}-\dfrac{\partial N}{\partial t}=0, & \dfrac{\partial L}{\partial y}-\dfrac{\partial M}{\partial x}+\dfrac{\partial Z}{\partial t}=0, \\ \dfrac{\partial L}{\partial x}+\dfrac{\partial M}{\partial y}+\dfrac{\partial N}{\partial z}=0, & \dfrac{\partial X}{\partial x}+\dfrac{\partial Y}{\partial y}+\dfrac{\partial Z}{\partial z}=0. \end{cases}$$

It seems surprising now that until 1907 it was not noticed that the best way to write these equations was the four-dimensional way already suggested by the form of Euler's equations (27). The explanation is that the equations were encumbered with matter and three-dimensional vector analysis. Moreover, the really fundamental things have a way of appearing to be simple once they have been stated by a genius, who was in this case Minkowski.

Another thing is almost as remarkable—the equations at which we have arrived are, except for some differences in sign, the same as the equations (20) of the Volterra theory for $n=4$, $r=2$. It remained for L. Hanni* to notice this fact in 1910. We have thus again in mathematical physics a generalization of the theory of analytic functions. The difference in sign is of extreme importance in physics. It may be stated that without this difference our world would have been dark; the minus sign that appears in the above quantities makes light possible. In this connection we might mention that if we eliminate all dependent functions but one from Maxwell's equations, we obtain, due to this difference in sign, instead of the Laplace equation that appears as a result of such elimination in the cases considered above, an equation of the type

$$(31)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 u}{\partial t^2}=0,$$

* L. Hanni, *Ueber den Zusammenhang zwischen den Cauchy-Riemannschen und den Maxwellschen Differentialgleichungen*, Tôhoku Mathematical Journal, vol. 5 (1910), pp. 142–175.

which differs from the Laplace equation by the minus sign before the last term. This is the so called *wave equation*; it received its name from the fact that it is often used in describing wave phenomena, especially in acoustics, and in optics in connection with what is called the wave theory of light. From the mathematical point of view the introduction by Maxwell of the electromagnetic theory of light may be considered as essentially the substitution for a theory based on one second-order partial differential equation of a theory based on a system of first-order equations (of which the second-order equation is a consequence). We want to emphasize this fact in view of an analogous situation to be mentioned later. We may mention also the fact that the main importance of this substitution seems to lie in its effect in introducing boundary conditions.

The lack of beauty due to the difference in sign may be remedied by the introduction of slight modifications in our notations. If we use instead of (21), the notations

$$(32)\quad \begin{cases} x = x_1, & y = x_2, & z = x_3, & t = ix_4, \\ X = if_{41}, & Y = if_{42}, & Z = if_{43}, & \\ L = f_{23}, & M = f_{31}, & N = f_{12}, & \end{cases}$$

and instead of (19) the relations

$$(33)\quad ir_{41} = f_{23},\ ir_{42} = f_{31},\ ir_{43} = f_{12},\ ir_{23} = f_{41},\ ir_{31} = f_{42},\ ir_{12} = f_{41},$$

the equations of Maxwell assume exactly the form (20). The introduction of these notations constitutes the second step toward the special theory of relativity.

4. *Second-Degree Quantities in Electrodynamics.* We noticed before that in Euler's hydrodynamic equations (27) the quantities u, v, w appear not directly, but as certain quadratic combinations; in the two- and three-dimensional cases written out above (28) and (29) the quantities subjected to differentiation are the elements of the matrices

$$(34)\quad \begin{pmatrix} \frac{1}{2}(u^2 - v^2) & uv \\ uv & \frac{1}{2}(v^2 - u^2) \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} \frac{1}{2}(u^2 - v^2 - w^2) & uv & uw \\ vu & \frac{1}{2}(v^2 - u^2 - w^2) & vw \\ wu & wv & \frac{1}{2}(w^2 - u^2 - v^2) \end{pmatrix}.$$

The differential equations (28) and (29) may be expressed by stating that the divergences of these matrices vanish. In four-space (special relativity theory) there also appears a matrix of the same nature which we write out in index notation:

$$M_{ij} = \rho(u_i u_j - \tfrac{1}{2}\delta_{ij} u_\alpha u_\beta). \tag{35}$$

In the electromagnetic theory analogous combinations of force components were introduced by Maxwell under the name of stress components. When we treat, in three-dimensional space, electric and magnetic forces separately, the stresses are

(36)

Electric Stresses			Magnetic Stresses		
$X^2 - e$	XY	XZ	$L^2 - m$	LM	LN
YX	$Y^2 - e$	YZ	ML	$M^2 - m$	MN
ZX	ZY	$Z^2 - e$	NL	NM	$N^2 - m$
with	$e = X^2 + Y^2 + Z^2$;		with	$m = L^2 + M^2 + N^2$.	

We may note that if we impose on the forces the equations (25), the divergence of the stress matrix vanishes, but we shall return to the question of differential relations later. We saw that the electric and magnetic forces are interrelated, and that the situation is best expressed in four-dimensional notations. Therefore there seems to be no reason for keeping the electric and magnetic stresses apart, and so we combine the above matrices by simply adding the corresponding elements together. We extend the three-rowed matrix thus obtained into a four-rowed matrix by adding a fourth row and a fourth column. We shall not write out this four-rowed matrix in full, nor shall we explain the physical meaning (which is a very important one) of the new additional quantities. Instead we shall write it in short notations which will bring out even more clearly the relation to the theory of analytic functions. We have here two surface vectors f and r represented by four-rowed matrices as our first-degree quantities. In terms of them, the four-rowed matrix of the stresses is

$$E = \tfrac{1}{2}(f^2 + r^2), \tag{37}$$

or, in index notation,

$$E_{ij} = \tfrac{1}{2} \sum (f_{i\alpha} f_{\alpha j} + r_{i\alpha} r_{\alpha j}). \tag{38}$$

Here f^2 and r^2 may be considered as matrices obtained from the

matrices f and r by squaring them in the manner employed in squaring a determinant. The expression obtained is a matrix which depends quadratically on the force matrix. The relation between the force matrices f and r on the one hand, and the matrix E on the other is analogous to the relation between the first-degree and the second-degree quantities in the theory of analytic functions. In some ways it reminds us of the quantity W. It is, like W, a matrix, and the analogy between W and E guided us in the formation of E, especially in the three-dimensional stage. As a finished product, however, E resembles more Q. In the first place, it possesses the same form (compare formulas (6)); in the second place, the relationship between E and Q appears very clearly when we ask ourselves whether the forces given by f are determined when E is given. Just as in the case of analytic functions, if (X, Y, Z) and (L, M, N) produce certain stresses, the quantities

$$(39)\begin{cases} X\cos\phi - L\sin\phi, & Y\cos\phi - M\sin\phi, & Z\cos\phi - N\sin\phi, \\ X\sin\phi + L\cos\phi, & Y\sin\phi + M\cos\phi, & Z\sin\phi + N\cos\phi, \end{cases}$$

where ϕ is an arbitrary function of x, y, z, and t, produce the same stresses. These formulas are analogous to (9).

We pass now to the consideration of differential equations satisfied by the stresses. The matrix E, as was the case with W in the case of an analytic function, satisfies, as a result of (f, r) satisfying Maxwell's equations, a set of linear differential equations which may be written in the form

$$(40) \qquad \frac{\partial E_{i\alpha}}{\partial x_\alpha} = 0,$$

and is strictly analogous to equations (7). This analogy appears more clearly when the last equations are written out in components. But these equations do *not* constitute a *sufficient* condition for a stress matrix to be derived from forces which satisfy Maxwell's equations. Subjecting to Maxwell's equations (30) the above expressions (39) for the forces corresponding to a given stress matrix, we obtain a set of eight equations on the derivatives of the heretofore arbitrary angle ϕ. The vanishing of the divergence of E (see equations (40)) is exactly the set of conditions for algebraic *compatibility* of these equations. In addition

to that, however, certain *integrability* conditions must be satisfied. They furnish another non-linear set of differential equations for E. We shall not write out these equations,* but for the purpose of further reference we shall put down the equations on the derivatives of the angle ϕ, which (derivatives) we shall denote by $\phi_1, \phi_2, \phi_3, \phi_4$:

$$
(41)\quad \begin{cases}
\dfrac{\partial Y}{\partial z} - \dfrac{\partial Z}{\partial y} - \dfrac{\partial L}{\partial t} - N\phi_2 + M\phi_3 - X\phi_4 = 0, \\
\dfrac{\partial Z}{\partial x} - \dfrac{\partial X}{\partial z} - \dfrac{\partial M}{\partial t} + N\phi_1 - L\phi_3 - Y\phi_4 = 0, \\
\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x} - \dfrac{\partial N}{\partial t} - M\phi_1 + L\phi_2 - Z\phi_4 = 0, \\
\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z} - L\phi_1 - M\phi_2 - N\phi_3 = 0, \\
\dfrac{\partial M}{\partial z} - \dfrac{\partial N}{\partial y} + \dfrac{\partial X}{\partial t} + Z\phi_2 - Y\phi_3 - L\phi_4 = 0, \\
\dfrac{\partial N}{\partial x} - \dfrac{\partial L}{\partial z} + \dfrac{\partial Y}{\partial t} - Z\phi_1 + X\phi_3 - M\phi_4 = 0, \\
\dfrac{\partial L}{\partial x} - \dfrac{\partial M}{\partial y} + \dfrac{\partial Z}{\partial t} + Y\phi_1 - X\phi_2 - N\phi_4 = 0, \\
\dfrac{\partial L}{\partial x} + \dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial z} + X\phi_1 + Y\phi_2 + Z\phi_3 = 0.
\end{cases}
$$

These equations are the analogs of (10).

In what precedes we have treated separately the components of the forces and the components of matter corresponding to the right hand sides and the left hand sides of the original equations of motion (23). Although in the description of the further fate of these quantities the theory of analytic functions—our guiding light—ceases to play a leading role, we do not want to abandon forces and matter without mentioning that they become happily reunited by simply adding together the corresponding elements of the matrices M_{ij} (35) and E_{ij} (38) into the elements of a ma-

* See G. Y. Rainich, Transactions of this Society, vol. 27 (1925), p. 129.

trix T_{ij}, and that the equations of motion may then be written simply by saying that the divergence of this new matrix is zero. The new equations of motion differ slightly from the old ones but seem to be in keeping with experiments. At this point another of the theories of pure mathematics becomes of extreme importance, namely, the theory of surfaces together with its generalization, the theory of curved spaces (riemannian geometry). A discussion of the interrelation between these two theories* of pure mathematics, and also the application of the second to physics, space does not permit to take up here. It will suffice to say that the identification of the matrix T_{ij} with a tensor appearing in the theory of curved spaces leads to the general relativity theory.†

5. *The Schroedinger Equation.* In its first stages of development, the quantum theory seemed to be very far removed from the continuous theories of which the theory of analytic functions is a model, but further development resulted in a surprising change which we shall sketch here.

We shall restrict our considerations to what the physicists call the hydrogen atom, or, more precisely, the spectrum of a hydrogen atom. We shall only mention the older theory (1913–1925) due to Bohr, which was based on the analogy with the solar system: a nucleus in the center and an electron going around it and (in the last stage of the theory) spinning at the same time. There were two points of difference: in the first place, not all orbits consistent with Newton's law or with Kepler's law, were permitted, but only a discrete set of orbits; in the second place, the different spectral lines were associated not with orbits but each line was associated with two orbits; the conception was that radiation is produced by an electron *jumping* from one orbit to the other.

We have here, just as at the beginning of mechanics, discontinuity in the distribution of matter, but it is accentuated by discontinuity in the set of permissible orbits and discontinuities in motion (jumps). The trend from discontinuity did not appear here until 1926 when the fundamental papers by Schroedinger

* See G. Y. Rainich, *Ueber die Analytischen Funktionen auf einer Minimalflaeche*, Mathematische Annalen, vol. 101 (1929), pp. 386–393.

† These questions were taken up in the Symposium lecture.

were published,* although it must be said that his work was not entirely independent of some ideas of his predecessors. From our point of view it is interesting to note that Schroedinger formulates his idea in the title of his papers as an attempt to interpret a certain set of numbers that appear in analyzing spectra as the set of characteristic values (Eigenwerte) of a *partial* differential equation. (This corresponds to the transition in mechanics from the theory of a moving point to the theory of the motion of a fluid (§3).) This partial differential equation in the simplest case is

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 - Vu + Eu = 0. \tag{42}$$

Here u is a complex-valued field function, V is a given function (in the special case we are considering, the potential of the field of forces in which the electron moves), E is an undetermined constant, and the units have been chosen in such a way as to simplify the coefficients. It is surprising that equations of this type have not been considered in connection with spectra before. In acoustics the equation

$$\Delta u + k^2 u = 0 \tag{43}$$

was used to determine the different states of vibration of a solid. The different characteristic values, that is, the values of k for which there exist solutions of this equation compatible with certain boundary conditions depending on the geometric configuration and the physical properties of the sounding body, were taken to correspond to the different pitches of sound produced. In quantum optics, however, we have instead of boundary conditions the additional condition that the solutions should be one-valued and continuous. This should be compared with the conditions imposed on the solutions of differential equations in the derivation of the power series as a general analytic function in (f), §1. The meaning of the potential V is that by changing it we obtain different sets of characteristic values, that is, different spectra, so that it takes over some functions of the boundary conditions, because in acoustics we ordinarily get different sets of characteristic values by taking different boundary conditions.

* E. Schroedinger, *Quantisierung als Eigenwertproblem.* First paper, Annalen der Physik, vol. 79 (1926), p. 361; second paper, ibid., p. 489; third paper, ibid., vol. 80 (1926), p. 437.

We shall now trace, following Sommerfeld,* the series of changes that the Schroedinger equation underwent, but we shall first call attention to the fact that the connection with the theories discussed heretofore is established by the fact of the appearance of the Laplace differential operator $\Delta u = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. To begin with, time does not appear in the equation, but we may introduce it by reversing the process by which we got rid of θ in the equation (11). We thus arrive at the equation

$$\Delta w - Vw + i \cdot (\partial/\partial t) w = 0, \tag{44}$$

which contains differentiation with respect to time but does not contain time explicitly in its coefficients; the substitution

$$w = ue^{iEt} \tag{45}$$

brings us back exactly to the equation we had before.

This substitution is of the same nature as (12) employed before while we were obtaining a power series as a solution of the Cauchy-Riemann equations. Its justification is here the same (the equation does not contain t explicitly)† and the result is analogous; the general solution of the equation (44) appears as a series whose terms are arbitrary constants multiplied by functions of the type $u_n e^{iE_n t}$, where u_n depend on x, y, z only, and E_n is a discrete set of constants.

The equation we are considering now is not quite satisfactory from the aesthetic point of view; its form is not in keeping with the principle of relativity which for our present purposes may be formulated as the principle of treating time and space coordinates alike; but it was found that the equation

$$\left(\frac{\partial}{\partial x} + \phi_1\right)^2 w + \left(\frac{\partial}{\partial y} + \phi_2\right)^2 w + \left(\frac{\partial}{\partial z} + \phi_3\right)^2 w - \left(\frac{\partial}{\partial t} + \phi_4\right)^2 w + A^2 w = 0, \tag{46}$$

* A. Sommerfeld, *Atombau und Spektrallinien*, Wellenmechanischer Ergaenzungsband, Braunschweig, 1929, pp. 119–120.

† This situation is a special case of a more general situation when the equation allows a one-parameter group of transformations (the one-parameter group of transformations being in this case $t' = t + p$); in the general case we also may use a similar device which results in obtaining a sequence of equations involv-

which is more satisfactory in form, gives results which are only slightly different from the results obtained from the preceding equation if we set $\phi_1=\phi_2=\phi_3=0$, $\phi_4=V$, and give to the constant A, which is the mass constant, an appropriate value. This improvement of the form without affecting much the numerical results might be compared with the transition from the original Euler equations to the equations mentioned at the end of §4.

We may say that we have now essentially a Laplace-like equation. More precisely, except for the potential ϕ_i and the mass term, we have the wave equation. Analogy suggests that we consider two things: second-degree quantities and first-order equations, of which the Laplace-like equation is a consequence.

Beginning with second-order quantities, we notice that w is a complex number and this suggests the consideration of $w\cdot\overline{w}$. If we take for w a particular solution corresponding to one term (13) of the power series in the case of analytic function, namely,

$$w = ue^{iEt},$$

we have $w\cdot\overline{w}=u\cdot\bar{u}$ and we see that t has dropped out. If we take a linear combination of such terms, however, let us say of two such terms,

$$w = a_1u_1e^{iE_1t} + a_2u_2e^{iE_2t}, \tag{47}$$

we get

$$w\cdot\overline{w} = a_1\bar{a}_1u_1\bar{u}_1 + a_2\bar{a}_2u_2\bar{u}_2 + a_1\bar{a}_2u_1\bar{u}_2e^{i(E_1-E_2)t} + a_2\bar{a}_1u_2\bar{u}_1e^{i(E_2-E_1)t}. \tag{48}$$

The E's are interpreted as energy values, and the set of discrete values of the E's corresponds to the set of orbits in the old Bohr theory. As a result of our forming the second-degree quantity $w\cdot\overline{w}$ we see then that we obtained in a natural way the fact that not the single orbits but pairs of orbits, not the separate energy values, but their differences appear in our formulas; or we may say, the transition from energy values to differences of energy values which in the Bohr theory led to the introduction of jumps, is achieved in the continuous theory by passing from

---

ing one less variable but involving a sequence of numbers, corresponding to the numbers k in the case of an analytic function; these numbers are used in physics as *quantum* numbers.

the linear quantities w to the quadratic quantities $w \cdot \bar{w}$. Another discontinuity is gone. It must be added that there remain apparently great difficulties in the interpretation of the coefficients a_i in the preceding formulas which temper the enthusiasm which was aroused in the minds of physicists when the situation described above was first discovered by Schroedinger.

6. *The Dirac Equations.* On one occasion before, in pursuing the analogy between analytic functions and wave mechanics, we reversed our steps; in analytic functions we used the substitution $w = Pe^{ik\theta}$ to eliminate a variable; in wave mechanics we used it to introduce a variable (t). We come now to another case of that nature. In analytic functions we arrive at the Laplace equation from a system of first-order equations. Here we already have a (generalized) Laplace equation and want to arrive from it at a system of first-order equations of which it is a consequence. Strictly speaking, we already have a solution of this problem. We saw that as a result of elimination of all but one component from Maxwell's system we arrive at a wave equation. It was also noted that the introduction of Maxwell's equations in optics may be considered as such a reversal from a second-order equation to a set of first-order equations. Dirac* achieved the same transition in wave mechanics, but he found another very remarkable set of first-order equations which lead to the wave equation. There are many attempts to substitute, for his equations, equations very closely related to Maxwell's equations, but it is not clear at present whether such an attempt will ever prove successful. We shall now explain Dirac's method. We strip the equation (46) of its unessential features, that is, the potentials and the mass-term. What remains we write in the form

$$(X^2 + Y^2 + Z^2 - T^2)w = 0, \tag{49}$$

where

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad Z = \frac{\partial}{\partial z}, \quad T = \frac{\partial}{\partial t},$$

and try to find a first-degree operator whose square is

* P. A. M. Dirac, *Quantum theory of the electron*, first paper, Proceedings of the Royal Society, (A), vol. 117 (1928), pp. 610–624; second paper, ibid., vol. 118 (1928), pp. 351–361.

$X^2+Y^2+Z^2-T^2$. We write it in the form $\alpha X+\beta Y+\gamma Z+\delta T$, and expressing our requirement, we get

$$\alpha^2=\beta^2=\gamma^2=-\delta^2=1,\quad \alpha\beta+\beta\alpha=0,\ \text{etc.} \tag{50}$$

In order to simplify our discussion we shall consider the two-dimensional case. The solution of the equations $\alpha^2=\beta^2=1$, $\alpha\beta+\beta\alpha=0$ in numbers is impossible, of course, but we can find matrices, viz.,

$$\alpha=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{51}$$

which satisfy these conditions. *Formally*, then, the equation $(\alpha X+\beta Y)w=0$ is equivalent to $(X^2+Y^2)w=0$, that is, to the equation of Laplace; or, at the least, the solutions of the first equation must satisfy the second. But how shall we interpret the operator with matrices? Among many roles in which the elements of a matrix appear, one of the most important is the role of coefficients of a linear transformation. Since we have two-rowed matrices, we consider two-component vectors (u, v). Now α transforms (u, v) into $(u, -v)$; β transforms (u, v) into (v, u). Our equation $(X\alpha+Y\beta)w=0$ gives then exactly the Cauchy-Riemann equations (1).

Returning to the four-dimensional case we would expect to arrive there at four-rowed matrices; this suggests four-component vectors, and that is what Dirac* and Darwin† found. After the equations have been obtained, we put back the potentials and the mass constant A where they belong. It is important to note that, without potentials, the transition from one second-order equation to four first-order equations would be only a change of form; it would not affect the physical results. It *does* make a great deal of difference, however, whether we put in the potentials before or after the transition. Dirac's theory fits the experimental facts still better than Schroedinger's. The transition from Schroedinger's theory to that of Dirac plays the same role as the introduction of spin in the Bohr theory. It was noted

* Dirac, loc. cit.

† C. G. Darwin, *The wave theory of the electron*, Proceedings of the Royal Society, (A), vol. 118 (1928), pp. 654–680; *On the magnetic moment of the electron*, ibid., vol. 120 (1928), pp. 621–631.

that the potentials play a role analogous to that played by boundary conditions in other cases, and it has been mentioned in connection with the introduction of Maxwell's equations in optics that the transition from second-order to first-order equations there was also important in connection with its effect on boundary conditions.

We now write out Dirac's equations essentially in the form given them by Darwin:

$$(52)\qquad \begin{cases} (p_4 + A)\psi_1 + (p_1 - p_2)\psi_4 + p_3\psi_3 = 0, \\ (p_4 + A)\psi_2 + (p_1 + ip_2)\psi_3 - p_3\psi_4 = 0, \\ (p_4 - A)\psi_3 + (p_1 - ip_2)\psi_2 + p_3\psi_1 = 0, \\ (p_4 - A)\psi_4 + (p_1 + ip_2)\psi_1 - p_3\psi_2 = 0, \end{cases}$$

where

$$p_4 = i\frac{\partial}{\partial t} + i\phi_4,\ p_1 = \frac{\partial}{\partial x} + \phi_1,\ p_2 = \frac{\partial}{\partial y} + \phi_2,\ p_3 = \frac{\partial}{\partial z} + \phi_3.$$

In these equations we split up the complex quantities into real and imaginary parts, using the notations

$$(53)\qquad \psi_1 = N + iP,\ \psi_2 = L + iM,\ \psi_3 = iZ - U,\ \psi_4 = iY - X.$$

Then the equations become

$$(54a)\qquad \begin{cases} \dfrac{\partial U}{\partial x} + \dfrac{\partial Z}{\partial y} - \dfrac{\partial Y}{\partial z} - \dfrac{\partial L}{\partial t} \\ \qquad + AM + Z\phi_1 - U\phi_2 - X\phi_2 + M\phi_4 = 0, \\ -\dfrac{\partial Z}{\partial x} + \dfrac{\partial U}{\partial y} + \dfrac{\partial X}{\partial z} - \dfrac{\partial M}{\partial t} \\ \qquad - AL + U\phi_1 + Z\phi_2 - Y\phi_3 - L\phi_4 = 0, \\ \dfrac{\partial Y}{\partial x} - \dfrac{\partial X}{\partial y} + \dfrac{\partial U}{\partial z} - \dfrac{\partial N}{\partial t} \\ \qquad - AP + X\phi_1 + Y\phi_2 + Z\phi_3 + P\phi_4 = 0, \\ \dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} + \dfrac{\partial Z}{\partial z} + \dfrac{\partial P}{\partial t} \\ \qquad + AN - Y\phi_1 + X\phi_2 - U\phi_3 + N\phi_4 = 0, \end{cases}$$

$$
(54b)\quad \begin{cases} \dfrac{\partial P}{\partial x}+\dfrac{\partial N}{\partial y}-\dfrac{\partial M}{\partial z}+\dfrac{\partial X}{\partial t} \\ \qquad +AY+N\phi_1-P\phi_2-L\phi_3-Y\phi_4=0, \\ -\dfrac{\partial N}{\partial x}+\dfrac{\partial P}{\partial y}+\dfrac{\partial L}{\partial z}+\dfrac{\partial Y}{\partial t} \\ \qquad -AX+P\phi_1+N\phi_2-M\phi_3+X\phi_4=0 \\ \dfrac{\partial M}{\partial x}-\dfrac{\partial L}{\partial y}+\dfrac{\partial P}{\partial z}+\dfrac{\partial Z}{\partial t} \\ \qquad +AU+L\phi_1+M\phi_2+N\phi_3-U\phi_4=0, \\ \dfrac{\partial L}{\partial x}+\dfrac{\partial M}{\partial y}+\dfrac{\partial N}{\partial z}+\dfrac{\partial U}{\partial t} \\ \qquad +AZ-M\phi_1+L\phi_2-F\phi_3-Z\phi_4=0. \end{cases}
$$

The first thing we notice about these equations is that they are a generalization of Maxwell's equations (25). In fact, if we make $U=P=0$, and also make all the potentials and the mass constant zero, we get exactly Maxwell's equations. Let us now keep U and P general but neglect the potentials and the mass constant, that is, neglect the last five terms in each equation. The resulting system is also a generalization of Maxwell's system, and therefore of the Cauchy-Riemann system (1). It has been investigated from this point of view by Iwanenko and Nikolski,* who consider all the eight components of Dirac as the components of a biquaternion, and, extending the investigation of Hanni,† obtain the equations as *integrability conditions*, that is, write certain integrals whose vanishing is equivalent to the original equations. It may be noted that these equations do *not* come under the Volterra theory.

A very interesting point about these equations is that although they are relativistically invariant, (that is, if we pass to another coordinate system, we may express exactly the same thing the equations tell us by a system of equations of the same type, with different values of the components, of course) the re-

* D. Iwanenko and K. Nikolski, Zeitschrift für Physik, vol. 63 (1930), p. 129.

† Loc. cit.

lations between the values of components in one system of coordinates and the values in another system, that is, the transformation formulas, are *not* of the type considered in ordinary tensor analysis. Much has been made of this circumstance. Weyl* connected it in a very beautiful way with the theory of representation of groups. Van der Waerden† invented a new tensor analysis, called spinor analysis, that takes care of the situation in a satisfactory way.

The situation takes a different aspect, however, when we introduce the *second-order quantities*. In the Schroedinger theory there was only one complex-valued component. We considered only one real-valued quadratic expression in it, namely its norm (see (48)). Now, as the result of the transition to first-order equations, we have many more components. Quite early in the development of the subject, in the first papers by Dirac and Darwin, a four-vector, two scalars, and a six-vector were introduced, whose components are quadratic in the ψ's, and are transformed according to the ordinary formulas of tensor calculus. Further investigation showed the existence of still another four-vector whose components are quadratic in the ψ's. As a result of the fact that all these components are expressed in terms of the four ψ's there exist between them many relations. Limiting our attention to the two four-vectors, we find that they are perpendicular and of equal length.‡ This reminds us of analytic functions; we ask ourselves whether the divergences of these vectors vanish and find that they *do*, in the case when the mass constant is zero. We have thus in the Dirac theory two equal and perpendicular vectors with vanishing divergences. The theory seems then to provide a method of studying such situations, which, as we saw before (beginning of §2), involve non-linear relations, by means of *linear* equations, because the Dirac equations are linear. The method is based on introducing auxiliary quantities in such a way that the quadratic relations are the result of the fact that the two vectors are expressed in terms of the same auxiliary quantities. Taking this hint we find

* H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, 1928, §§25 and 39.

† B. L. van der Waerden, Göttinger Nachrichten, 1929, p. 100.

‡ O. Laporte and G. E. Uhlenbeck, Physical Review, vol. 37 (1931), pp. 1396 and 1553.

without difficulty that the components of two equal and perpendicular vectors in ordinary three-space (14) may be expressed as follows:

$$(55)\quad \begin{cases} f_1 = YZ - XU, & r_1 = \tfrac{1}{2}(X^2 + Z^2 - Y^2 - U^2), \\ f_2 = \tfrac{1}{2}(X^2 + Y^2 - Z^2 - U^2), & r_2 = XU + YZ, \\ f_3 = XZ + YU, & r_3 = ZU - XY. \end{cases}$$

If we subject these vectors to the condition (16) that their divergences should vanish, we obtain for the auxiliary quantities, X, Y, etc., a system of equations, which, although not linear, is a consequence of the linear system:

$$(56)\quad \begin{aligned} &\frac{\partial X}{\partial x} + \frac{\partial U}{\partial y} - \frac{\partial Y}{\partial z} = 0, \quad \frac{\partial Y}{\partial x} - \frac{\partial Z}{\partial y} + \frac{\partial X}{\partial z} = 0, \\ &\frac{\partial Z}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial U}{\partial z} = 0, \quad \frac{\partial U}{\partial x} - \frac{\partial X}{\partial y} - \frac{\partial Z}{\partial z} = 0, \end{aligned}$$

which in many ways is similar to the Dirac system with potentials and mass constant neglected.

In conclusion, a word concerning the potentials. As in the other cases in which we considered second-degree quantities, we may ask the questions whether the first-degree quantities are determined by the second-degree quantities, and what differential conditions are imposed on the second-degree quantities by the differential equations to which the first-order quantities are subjected. The answer to the first question is that the first-degree quantities are not completely determined by the second-degree quantities, but are determined to within a variable angle, just as (u, v) are not determined by u^2+v^2, or r and f by E. In answering the second question, we have to eliminate therefore this variable angle, just as in the cases we had before. Without entering into details we may state that if we start with Dirac's equations without potentials we obtain in a way analogous to that by which we obtained (10) and (41), a system of exactly the form of the Dirac equations with potentials, except that the ϕ's are the derivatives of the arbitrary angle.

THE UNIVERSITY OF MICHIGAN

APPLICATION OF BOOLEAN ALGEBRA TO PROVING CONSISTENCY AND INDEPENDENCE OF POSTULATES*

BY B. A. BERNSTEIN

1. *Object.* The usual procedure in proving the consistency of a set of postulates is to find such an interpretation of the undefined ideas involved in the postulates as to make all the postulates true propositions. The procedure in proving the independence of a particular postulate from the rest of the set is to find such an interpretation of the undefined ideas as to make that postulate, and only that one, false. But there exists no *method* of finding these interpretations. The object of this paper is to show how Boolean algebra partially provides such a method.

2. *General Procedure.* The postulates of a mathematical science are conditions on the classes, operations, and relations constituting the undefined ideas of the science. There is an infinitude of Boolean algebras.† In each of these algebras there is, for every positive integer n, the general n-ary operation

$$\text{(I)} \qquad Axy\cdots t + Bxy\cdots t' + \cdots + Lx'y'\cdots t'‡$$

and the general n-adic relation

$$\text{(II)} \qquad Axy\cdots t + Bxy\cdots t' + \cdots + Lx'y'\cdots t' = 0.$$

The procedure constituting a partial method of proving the consistency and the independence of a set of postulates consists, in the case of a consistency proof, in obtaining the conditions on the *discriminants* of (I) and (II) which will make (I) and (II) satisfy all the given postulates; if these Boolean conditions are consistent,§ the given postulates are consistent. In the case of

* Presented to the Society, June 20, 1929.

† There is a Boolean algebra the cardinal number of whose elements is that of the continuum; one whose elements are denumerably infinite; and one for every positive integer n, the number of whose elements is 2^n.

‡ The usual Boolean notation is used, with a' denoting the negative of a.

§ The determination of the consistency of a set of Boolean relations is, as is well known, always possible. The vanishing of the product of the discriminants of the single relation equivalent to the set is the necessary and sufficient condition for this consistency.

proof of the independence of a postulate P from the other postulates, the procedure consists in obtaining the conditions on the discriminants of (I) and (II) which will make (I) and (II) contradict P and not contradict any of the other postulates; if these Boolean conditions are consistent, postulate P is independent of the other postulates. If in the case of a consistency proof the resulting Boolean conditions are inconsistent, the consistency of the postulates remains undetermined. Similarly for the case of an independence proof. It is this possibility of an inconsistent set of resulting Boolean conditions which makes my procedure only a partial method for determining consistency and independence.

It is to be observed that the method here outlined does not require the actual finding of the desired Boolean consistency and independence systems, though such systems, when the Boolean conditions are consistent, can always be found.* The consistency of the resulting Boolean conditions proves that a desired Boolean proof-system *can* be found, and this possibility is sufficient for our purpose.

It is to be observed also that the conditions on the discriminants of (I) and (II) corresponding to a given postulate may be used in the investigation of consistency and independence of any set of propositions involving that postulate. At the same time, these conditions give valuable information concerning the operations and relations of Boolean algebra.

The proposed procedure of investigating consistency and independence of postulates thus gives results that are generally of far wider application than the results obtained by the usual methodless procedure.

The method outlined above will be made clearer by applying it to a set of postulates for abelian groups and to a set of postulates for series.

---

* The desired proof-system is simply the system in which the elements are the elements of the Boolean algebra employed, and the operations and relations are the operations and relations of forms (I) and (II) *with the discriminants satisfying the conditions in question*. Thus, the Boolean consistency system for postulates P_1-P_3 of §3 below, is the system $(K, \circ)$ in which K is the class of elements of any Boolean algebra and $a \circ b$ is the operation (α) with the discriminants satisfying condition (4).

3. *Application to Postulates for Abelian Groups.* An *abelian group* is a class K containing a binary operation $\circ$ satisfying postulates P_1-P_3 following:*

P_1. $a \circ (b \circ c) = (a \circ b) \circ c$, whenever the indicated elements belong to K.

P_2. For any two elements a, b of K there is an element x of K such that $a \circ x = b$.

P_3. $a \circ b = b \circ a$, whenever the indicated elements belong to K.

To investigate the consistency and the independence of P_1-P_3, consider the general binary Boolean operation

$$(\alpha) \qquad Aab + Bab' + Ca'b + Da'b'.$$

The necessary and sufficient conditions that (α) satisfy P_1, P_2, P_3 are, respectively:†

$$(1) \qquad D = AD, \quad (BC' + B'C)(AD + A'D') = 0,$$

$$(2) \qquad B = A', \quad C = D',$$

$$(3) \qquad B = C.$$

From $(1)-(3)$, it is seen that (α) will contradict P_1, P_2, P_3 if, respectively,

$$(1)' \qquad DA' = 1,$$

$$(2)' \qquad AB + A'B' = 1,$$

$$(3)' \qquad BC' + B'C = 1.$$

Relations $(1)-(3)$ are together equivalent to the relation

$$(4) \quad f(A, B, C, D) \equiv DA' + AB + A'B' + CD + C'D' + BC' + B'C = 0.$$

Relations $(1)'$, (2), (3) are together equivalent to

$$(5) \qquad f_1(A, B, C, D) \equiv 1 = 0.$$

Relations (1), $(2)'$, (3) together are equivalent to

$$(6) \quad f_2(A, B, C, D) \equiv DA' + AB' + A'B + BC' + B'C = 0.$$

---

* See Transactions of this Society, vol. 4 (1903), p. 27.

† See Transactions of this Society, vol. 26 (1924), pp. 171–173.

Finally, relations (1), (2), (3)′ together are equivalent to

$$(7) \qquad f_3(A, B, C, D) \equiv A' + B + C' + D = 0.$$

Since the discriminant $f(0, 1, 1, 0)$ vanishes, condition (4) is possible. Hence, postulates $P_1 - P_3$ are consistent.

Condition (5) is impossible, and the question of the independence of postulate P_1 remains unsettled.*

Since $f_2(1, 1, 1, 1) = 0$, condition (6) is possible. Hence P_2 is independent of the other postulates.

Finally, since $f_3(1, 0, 1, 0) = 0$, condition (7) is possible. Hence, P_3 is independent of the other postulates.

4. *Application to Postulates for Series.*† The elements a, b, $c, \cdots$ of a class K form a *series* with respect to a dyadic relation S, if they satisfy the postulates $Q_1 - Q_3$ following:

Q_1. *If $a \neq b$, then either aSb or bSa.*

Q_2. *If aSb, then $a \neq b$.*

Q_3. *If aSb and bSc, then aSc.*

Consider the general dyadic Boolean relation

$$(\beta) \qquad Aab + Bab' + Ca'b + Da'b' = 0.$$

This relation, I have shown, can be a serial relation only if the Boolean algebra is a two-element one. Let (β), then, be the general dyadic relation in a two-element Boolean algebra.

The necessary and sufficient conditions that (β) satisfy Q_1 and Q_2 are, respectively:

$$\text{(i)} \qquad BC = 0,$$

$$\text{(ii)} \qquad AD = 1.$$

The necessary and sufficient condition that (β) satisfy Q_3 non-vacuously is

$$\text{(iii)} \qquad (A + D)B'C' = 0, \quad AD = 0.$$

The necessary and sufficient condition that (β) satisfy Q_3 vacuously is

$$\text{(iv)} \qquad AD(B + C) = 1.$$

---

* That the denial of (1) is inconsistent with (2) and (3) may also be seen by observing that (2) and (3) imply (1).

† For the results through condition (iv) below, see this Bulletin, vol. 32 (1926), pp. 523–524.

From (i)–(iii), relation (β) will contradict Q_1, Q_2, Q_3 if, respectively:

(i)′ $BC = 1$,

(ii)′ $AD = 0$,

(iii)′ $AD = 1$.

Conditions (i)–(iii) are together equivalent to

(v) $1 = 0$.

Conditions (i), (ii), (iv) are together equivalent to

(vi) $F(A, B, C, D) \equiv BC + A' + D' + B'C' = 0$.

Conditions (i)′, (ii), (iv) are together equivalent to

(vii) $F_1(A, B, C, D) \equiv B' + C' + A' + D' = 0$.

Conditions (i), (ii)′, (iii) are together equivalent to

(viii) $F_2(A, B, C, D) \equiv BC + AD + (A + D)B'C' = 0$.*

Finally, conditions (i), (ii), (iii)′ are together equivalent to

(ix) $F_3(A, B, C, D) \equiv BC + A' + D' = 0$.

Relation (v) is impossible. Hence, in a two-element Boolean algebra there is no relation (β) which satisfies $Q_1 - Q_3$ non-vacuously, and the question of the consistency of $Q_1 - Q_3$ remains unsettled.

But relation (vi) is possible, since the discriminant $F(1, 1, 0, 1)$ vanishes. Hence, postulates $Q_1 - Q_3$ are consistent.

Relation (vii) is possible, since $F_1(1, 1, 1, 1) = 0$. Hence postulate Q_1 is independent of the other postulates.

Relation (viii) is possible, since $F_2(0, 0, 0, 0) = 0$. Hence Q_2 is independent of the other postulates.

Finally, relation (ix) is possible, since $F_3(1, 0, 0, 1) = 0$. Hence, Q_3 is independent of the other postulates.

THE UNIVERSITY OF CALIFORNIA

* If (iv) had been used here instead of (iii), the resulting condition would have been $1 = 0$, from which no conclusion could have been drawn with regard to the independence of Q_2.

A NON-DENSE PLANE CONTINUUM†

BY J. H. ROBERTS

The author has shown‡ that if M denotes a square plus its interior in a plane S, then there exists an upper semi-continuous collection§ G of mutually exclusive non-degenerate subcontinua of M filling up M and such that G is homeomorphic with M. The present paper gives a continuum M *which contains no domain* yet which has the above property.

Let I denote the interior of a square J in a plane S. Let G be an upper semi-continuous collection of mutually exclusive non-degenerate continua filling $J+I$ such that G is homeomorphic with $J+I$. Since no element of G separates S it follows‖ that if S' denotes the collection consisting of the elements of G and the *points* of S not belonging to any element of G, then S' corresponds to S under a continuous one to one correspondence U, and G corresponds to $J+I$. Let G^* denote the subcollection containing every element of G which contains a point of J. The set G^* is a simple closed curve. Moreover every element of G^* has in common with J either an arc or a single point. Then there exists¶ a continuous one to one correspondence U_1 between G^*

---

† Presented to the Society, August 30, 1929.

‡ *On a problem of C. Kuratowski concerning upper semi-continuous collections*, Fundamenta Mathematicae, vol. 14 (1929), pp. 96–102.

§ For a definition of this term, and of the notion *limit element*, see R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.

‖ See R. L. Moore, loc. cit., Theorem 22.

¶ This may be shown as follows. Let $s_1, s_2, s_3, \cdots$ denote the maximal arcs which are subsets of J and which belong to some element of G^*. For each i and j $(i \neq j)$ the set $s_i \cdot s_j$ is vacuous. Let $v(s_i)$ denote the length of the interval s_i, and $v(J)$ the length of J. Suppose first that $v(J) - \sum_{i=1}^{\infty} v(s_i)$ is a positive number e and let d be a positive number less than e. A sequence of segments $t_1, t_2, t_3, \cdots$ can be defined inductively so that (1) for each i there is a j such that t_j contains s_i, (2) no two of the segments $t_1, t_2, t_3, \cdots$ have any point in common, and (3) $\sum_{i=1}^{\infty} v(t_i)$ is less than $v(J) - d$. Since the point set $J - \sum_{i=1}^{\infty} t_i$ has positive measure it is uncountable. Now the curve J can be transformed into itself in such a way that the sum of the lengths of the images of the intervals $s_1, s_2, s_3, \cdots$ is less than the length of J. Hence in any case there is a set of segments

and J such that for uncountably many points x of J it is true that $U_1(x)$ (the element of G^* corresponding to the point x) contains x. The correspondence U_1 can† be extended so that there results a continuous one to one correspondence π between G and $J+I$ such that for uncountably many points P of J it is true that $\pi(P)$ contains P.

Now only a countable number of elements of G contain domains. Let H denote $J+I$ and let x denote any point of H. Let $C_1(x)$ denote x. Let $C_2(x)$ be the continuum $\pi(x)$. Let $C_3(x)$ denote the sum of all continua $\pi(y)$ for all points y of $C_2(x)$. In general let $C_{n+1}(x)$ denote the sum of all elements $\pi(y)$ for every point y of $C_n(x)$. If x and y are distinct points of H, then $C_n(x)$ and $C_n(y)$ are mutually exclusive continua. Hence, for each n, the set of all points x such that $C_n(x)$ contains a domain is countable. Hence there exist two points P_1 and P_2 of J, and a simple continuous arc T from P_1 to P_2, such that if x is any point of T then, for every n, $C_n(x)$ contains no domain, and $\pi(P_i)$ contains $P_i (i=1, 2)$.

There exists an infinite set of simple closed curves J_1, J_2, J_3, $\cdots$ such that (1) $J_1=J$, (2) $J_1 \cdot J_2$ is the point P_2, and in general, for each i, $J_i \cdot J_{i+1}$ is a single point P_{i+1}, and $J_i \cdot J_{i+k} = 0 (k>1)$, (3) no point of J_k lies within J_i, and (4) only a finite number of the curves J_1, J_2, J_3, $\cdots$ have points within any circle. For each i let π_i be a continuous transformation throwing

---

$l_1, l_2, l_3, \cdots$ satisfying (1) and (2) above, and such that $J-\sum_{i=1}^{\infty} l_i$ is uncountable, and indeed if $v(J)-\sum_{i}^{\infty} v(s_i)$ is a positive number e, then the segments $l_1, l_2, l_3, \cdots$ can be so chosen that the measure of the set $J-\sum_{i=1}^{\infty} l_i$ is as near e as we please. If P is any point of $J-\sum_{i=1}^{\infty} l_i$, then let $C(P)$ denote P. Consider every interval s_j that is a subset of l_i as an element, and every other point of $\bar{l}_i$ as an element. Then the collection of elements so obtained is an arc, and can be made to correspond to the arc $\bar{l}_i$ of J. Thus if T denotes the collection of intervals $s_1, s_2, s_3, \cdots$ and all other points of J, then there exists a correspondence C such that (1) $C(T)=J$ and (2) for uncountably many points P of J the point P is an element of T, and $C(P)=P$. But if x is any element of T, then there is a continuum g_x of G^* containing x, and the correspondence D throwing g_x into x, for every element x of T, is continuous. Then if g is an element of G^* the correspondence throwing g into $C[D(g)]$ is a continuous one to one correspondence between G^* and J and satisfies the required conditions.

† See Schoenflies, *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), pp. 286–328. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), pp. 529–531.

H into J_i plus its interior in such a way that $\pi_i(P_1)=P_i$, and $\pi_i(P_2)=P_{i+1}$. For each n let E_n denote the sum of all continua $C_n(x)$ for every point x of the arc T. Note that E_{n+1} can be obtained by adding together all continua $\pi(y)$ for all points y of E_n. Let M_n denote $\pi_n(E_n)$ and let M denote $M_1+M_2+M_3+\cdots$. Then M is the continuum desired.

Since no one of the sets M_1, M_2, M_3, $\cdots$ contains a domain, the continuum M contains no domain. Let R denote $P_1+P_2+P_3+\cdots$ and let x denote any point of $M-R$ belonging to $M_i(i>1)$. Let y_x be the point of H such that $\pi_i(y_x)=x$. Let g_{y_x} denote the element of G corresponding, under π, to the point y_x, and let h_x denote the continuum $\pi_i(g_{y_x})$. In case h_x does not contain P_i or P_{i+1}, then let k_x denote h_x. For each i $(i>2)$ there is a point x_{i-1} of M_{i-1}, and a point $\bar{x}_i$ of M_i such that both the sets $h_{x_{i-1}}$ and h_{x_i} contain P_i. Let k_{P_i} denote $h_{x_{i-1}}+h_{\bar{x}_i}$. For some point x of M_2 the set h_x contains P_2. Let k_{P_2} denote h_x plus the arc T. Then M is the sum of the elements of an upper semi-continuous collection G' of mutually exclusive continua, every element of G' being a continuum k_Q for some point Q of M. The elements k_{P_3}, k_{P_4}, k_{P_5}, $\cdots$ are each homeomorphic with the sum of two elements of G and every other element of G' except k_{P_2} is homeomorphic with some one element of G. Every element of G' is a nondegenerate continuum. For each i $(i>1)$ let G_i' denote the collection of all elements k_P of G' for all points P of M_i. Then G_2' is homeomorphic with the arc M_1, G_3' is homeomorphic with M_2, G_4' with M_3, and so on indefinitely. Moreover, G_i' and G_{i+1}' have exactly one element in common, which corresponds to the common point of M_{i-1} and $M_i(i>1)$. Thus G and M are homeomorphic.

A bounded continuum with the same property may be obtained if condition (4) satisfied by the curves J_1, J_2, J_3, $\cdots$ is replaced by the following: (4) J_1, J_2, J_3, $\cdots$ is a contracting sequence having P_1 as sequential limit point. A continuum so obtained will have exactly two complementary domains in the plane S.

THE UNIVERSITY OF TEXAS

ON A THEOREM OF VON STERNECK

BY D. H. LEHMER*

1. *Introduction.* In 1894 von Sterneck† gave an interesting theorem which is essentially as follows.

THEOREM 1. *Let f and g be any numerical functions and consider the function h defined for all positive integers n by the equation*

$$h(n) = \sum_{n} f(i)g(j),$$

the summation extending over all (i, j) whose L.C.M. is n. Next write $F(n) = \sum f(\delta)$, $G(n) = \sum g(\delta)$, $H(n) = \sum h(\delta)$, where δ ranges over all the divisors of n. Then $F(n)G(n) = H(n)$.

It is the purpose of this note to supply the background for this theorem and to give a very simple proof of a theorem including among other special cases von Sterneck's theorem.

In a recent paper‡ the author has considered the class of functions $\psi(x, y)$ which are subject to the following conditions or postulates.

POSTULATE A. *If x and y are positive integers, so is $\psi(x, y)$.*

POSTULATE I. *For each integer $n > 0$, the equation $\psi(x, y) = n$ has but a finite number of solutions (x, y).*

POSTULATE II. $\psi(x, y) = \psi(y, x)$.

POSTULATE III. $\psi(\psi(x, y), z) = \psi(x, \psi(y, z))$.

POSTULATE IV. $\psi(x, 1) = n$, *implies* $x = n$.

The solutions x of $\psi(x, n) = n$ are called the *associates* of n. For the purposes of this paper we add the following postulate.

POSTULATE B. *$\psi(x, y)$ is not an associate of n, when neither x nor y are associates of n.*

---

* National Research Fellow.

† Monatshefte, vol. 5 (1894), p. 265. See also Dickson, *History of the Theory of Numbers*, vol. 1, p. 152. Von Sterneck's theorem is stated in terms of k independent functions instead of 2, as above. There is very little trouble in extending Theorem 1 and also Theorem 4 to this more general case. To avoid the consequent typographical complications, however, we leave this part to the reader.

‡ Transactions of this Society, vol. 33 (1931).

2. *Examples of ψ-Functions.* To each such ψ-function there corresponds a type of arithmetic of numerical functions. Only two ψ-functions are well known, and these lead to familiar branches of the theory of numbers. These functions are $x+y-1$ and xy. Much less familiar is the case* $\psi(x, y) = [x, y]$, the L.C.M. of x and y. Perhaps the simplest example of all is $\psi(x, y) = \max(x, y)$. This case seems to have attracted no attention.

THEOREM 2. *The most general polynomial satisfying the above postulates is*

$$Axy - (A - 1)(x + y - 1),$$

where A is a non-negative integer.

PROOF. Any polynomial $P(x, y)$ must be symmetric by Postulate III. If d is the degree of this polynomial in each variable, then comparing the degrees in x of both sides of

$$P(P(x, y), z) = P(x, P(y, z)),$$

we have $d^2 = d$. Now $d \neq 0$ because $\psi =$ constant fails to satisfy Postulate IV. Hence $d = 1$, and $P(x, y)$ is bilinear. Writing

$$P(x, y) = Axy + B(x + y) + C,$$

and using Postulate IV we find that $B = 1 - A = -C$. Hence

$$P(x, y) = Axy - (A - 1)(x + y - 1).$$

This polynomial satisfies the postulates if and only if A is an integer $\geqq 0$.

3. *Factorable ψ-Functions.* Given a ψ-function, the fundamental theorem of arithmetic enables us to write down as many other ψ-functions as we please. The method is as follows. Every positive integer n may be uniquely described by giving the exponents n_v to which the vth prime p_v appears as a factor of $n = \prod p_v^{n_v}$ for $v = 1, 2, 3, \cdots$. Let $\psi_1(x, y)$ be given. We define $\psi_2(x, y)$ by

$$(1) \qquad \psi_2(x, y) = \psi_2\Big(\prod p_v^{x_v}, \prod p_v^{y_v}\Big) = \prod p_v^{\psi_1(x_v+1, y_v+1)-1}.$$

* Von Sterneck seems to have first used the function L.C.M. in this connection. The fundamentals of the L.C.M. calculus are developed in a recent paper, American Journal of Mathematics, vol. 53 (1931). See also a paper by E. T. Bell, this Bulletin, vol. 37 (1931), p. 85.

Such a ψ-function is called factorable. As may be verified, ψ_2 satisfies the above postulates provided of course ψ_1 does also. Of the functions mentioned in §2, only two are factorable, namely xy and $[x, y]$, and these are derived from $x+y-1$ and max (x, y) respectively. These factorable functions in turn produce other factorable functions, namely, $\prod p_v^{x_v+y_v+x_v y_v}$* and $\prod p_v^{[x_v+1,\, y_v+1]-1}$ and so on. The general polynomial ψ-function gives rise to $\prod p_v^{x_v+y_v+Ax_v y_v}$. The process may be reversed and carried back until a non-factorable function is reached.

Instead of representing n as a product of powers of primes we may use its digits when written to the base 10, for example. Thus $n=\sum_{r=0} n_r 10^r$, $0 \leqq n_r \leqq 9$. From the function $\psi_1(x, y) = \max(x, y)$ we derive

$$\psi_2(x+1, y+1) = 1 + \sum_{r=0} \max(x_r, y_r) 10^r.$$

4. *The Main Theorem.* We shall first prove the following theorem.

THEOREM 3. *$\psi(x, y)$ is an associate of n if and only if both x and y are associates of n.*

PROOF. Suppose first that both x and y are associates of n. Then by Postulate III,

$$\psi(\psi(x, y), n) = \psi(x, \psi(y, n)) = \psi(x, n) = n.$$

Hence $\psi(x, y)$ is an associate of n. Next suppose that one variable, say y, is, and x is not an associate of n. Then we can write

$$\psi(\psi(x, y), n) = \psi(x, \psi(y, n)) = \psi(x, n) \neq n.$$

Hence $\psi(x, y)$ is, in this case, not an associate of n. Postulate B now completes the proof of the theorem. We now pass to the proof of the main theorem.

THEOREM 4. *Let f and g be any numerical functions and consider the function h defined for all positive integers n by the equation*

$$h(n) = \sum_n f(i)g(j), \tag{2}$$

* This function is not too complicated to give an interesting type of arithmetic, some details of which are given in the Transactions of this Society. The value of this function depends upon the multiplicative properties of exponents of the primes dividing x and y.

the summation extending over all (i, j) for which $\psi(i, j) = n$. Next write $F(n) = \sum f(a)$, $G(n) = \sum g(a)$, $H(n) = \sum h(a)$, where a ranges over the associates of n. Then $F(n)G(n) = H(n)$.

PROOF. Let the function $\epsilon_n(m)$ have the value 1 or 0 according as m is or is not an associate of n. Then if x and y are any positive integers,

$$\epsilon_n(x)\epsilon_n(y) = \epsilon_n(\psi(x, y)). \tag{3}$$

In fact this follows at once from Theorem 3. Hence we can write

$$\begin{aligned} F(n)G(n) &= \sum f(a) \sum g(a) = \sum_{x=1} f(x)\epsilon_n(x) \sum_{y=1} g(y)\epsilon_n(y) \\ &= \sum f(x)g(y)\epsilon_n(\psi(x, y)) \\ &= \sum_{m=1} \epsilon_n(m) \sum f(x)g(y), \qquad (\psi(x, y) = m), \\ &= \sum_{m=1} \epsilon_n(m)h(m) = \sum h(a) = H(n), \end{aligned}$$

which is the theorem.

5. *Examples of Theorem* 4. In case $\psi(x, y)$ is a polynomial $P(x, y)$, the preceding theorem is trivial. In fact the equation $P(x, n) = n$ is linear in x and hence has the single solution $x = 1$ required by Postulate IV. That is, 1 is the only associate of n. Theorem 4, in this case, simply states that $f(1)g(1) = h(1)$, which is (2) written for $n = 1$. It may be shown that if ψ_1 is such that 1 is the only associate of n, $(n = 1, 2, 3, \cdots)$, then the same is true of the factorable function ψ_2 defined by (1). Hence Theorem 4 is trivial for a ψ-function which is derived in this way from a polynomial.

Let $\psi(x, y) = \max(x, y)$. Then the associates of n are $1, 2, 3, \cdots, n$. Hence if we write

$$F(n) = \sum_{k=1}^{n} f(k), \quad G(n) = \sum_{k=1}^{n} g(k),$$

Theorem 4 asserts that

$$\sum_{k=1}^{n} \{f(k)G(k) + g(k)F(k) - f(k)g(k)\} = F(n)G(n).$$

In case $\psi(x, y) = [x, y]$, the associates of n are its divisors. Theorem 4 then becomes von Sterneck's Theorem 1.

STANFORD UNIVERSITY

A NOTE ON CYCLIC ALGEBRAS OF ORDER SIXTEEN

BY A. A. ALBERT*

1. *Introduction.* In a recent paper† I considered cyclic (Dickson) algebras of order sixteen generated by a cyclic quartic field Z and a quantity γ in the reference field F. It was proved there that if the algebra A were a division algebra and if γ^2 were the norm of a quantity of Z, so that the Wedderburn *norm condition* would not be satisfied, then A would be the direct product of two generalized quaternion algebras. It was not proved, however, that such division algebras existed.

R. Brauer has recently‡ proved that there exist normal division algebras which are direct products of two generalized quaternion algebras. In the present note I give an example of a *cyclic* algebra over the Brauer reference field for which the *norm condition* is not satisfied, therefore completing the theory of the previous paper.

2. *The Example.* Let $F=R(\xi, \eta)$, where ξ and η are indeterminates and R is the field of all rational numbers. This is the reference field of the algebras of Brauer. We shall use the notations of my paper (loc. cit.), Theorem 3. It was proved there that a necessary and sufficient condition that a direct product of two generalized quaternion algebras over F be a division algebra is that the connected form

$$\tau x_1^2 + \sigma x_2^2 - \sigma\tau x_3^2 - (\gamma x_4^2 + \rho x_5^2 - \rho\gamma x_6^2), \tag{1}$$

in the variables $x_1, \cdots, x_6$ in F be not a null form. We shall take

$$\sigma = -2\xi^3, \quad \rho = \eta, \quad \gamma = -1, \quad \tau = \alpha, \tag{2}$$

where α is a rational number not the square of a rational number.

Suppose that $\alpha=\nu^2$, where ν is in F. Then we may write

$$\nu = bc^{-1},$$

---

* Presented to the Society, September 9, 1931.

† This Bulletin, vol. 37 (1931), pp. 301–312.

‡ Mathematische Zeitschrift, vol. 31 (1930), §5. Brauer's example is that of an algebra not necessarily a cyclic algebra and it would probably be difficult to prove it cyclic even if this were the case. Also his proof that the algebra is a division algebra is essentially different from ours.

where $b=b(\xi, \eta)$ and $c=c(\xi, \eta)$ are polynomials in ξ and η with coefficients in R such that the greatest common divisors of b and c are rational numbers. Then

$$b^2 = \alpha c^2, \quad b_1^2 = \alpha c_1^2,$$

where $b_1=b(\xi, 0)$, $c_1=c(\xi, 0)$. The coefficient of the highest power of ξ in b_1^2 is evidently a rational square, that in $c_1^2\alpha$ not such a square, since α is not a rational square. Hence $b_1^2=c_1^2\alpha$ implies that $b_1=c_1=0$ and b and c have the common factor η, a contradiction.

Suppose that (1) vanished for $x_1, \cdots, x_6$ not all zero and in F. Evidently we could take $x_1, \cdots, x_6$ to be polynomials in ξ and η with rational coefficients and with greatest common divisor a rational number. We thus write

$$x_i = x_i(\xi, \eta),$$

with rational coefficients and having no factor in common. Equation (1) becomes

$$\alpha x_1^2 - 2\xi^3 x_2^2 + 2\alpha\xi^3 x_3^2 = -x_4^2 + \eta(x_5^2 + x_6^2), \tag{3}$$

which must be satisfied identically in ξ and η so that it must be satisfied when we replace η by zero. Let $x_i(\xi, 0)=y_i$ and let the highest power of ξ occurring in y_i be ξ^{r_i}, its coefficient being λ_i where $\lambda_i=0$ if and only if $y_i=0$. Then (3) becomes

$$y_4^2 + \alpha y_1^2 = 2\xi^3(y_2^2 - \alpha y_3^2), \tag{4}$$

identically in ξ. According as $r_4>r_1$, $r_4<r_1$, $r_4=r_1$, the term of highest degree in $y_4^2+\alpha y_1^2$ is

$$\lambda_4^2\xi^{2r_4}, \quad \alpha\lambda_1^2\xi^{2r_1}, \quad (\lambda_4^2 + \alpha\lambda_1^2)\xi^{2r_1},$$

an even power, and is zero if and only if $y_1=y_4=0$. Similarly the possible terms of highest degree in $2\xi^3(y_2^2-\alpha y_3^2)$ are

$$2\lambda_2^2\xi^{2r_2+3}, \quad -2\alpha\lambda_3^2\xi^{2r_3+3}, \quad 2(\lambda_2^2 - \alpha\lambda_3^2)\xi^{2r_2+3},$$

so that (3) and its consequence (4) imply that a polynomial whose degree is even is equal to a polynomial whose degree is odd. This is possible only when both polynomials are zero, so that $\lambda_4=\lambda_1=\lambda_2=\lambda_3=0$ and hence $y_1=y_2=y_3=y_4=0$. But then $x_1, \cdots, x_4$ are divisible by η, so that we may write

$$x_i = \eta z_i, \qquad (i = 1, \cdots, 4),$$

and obtain

$$\eta^2(z_1^2 - 2\xi^3 z_2^2 + 2\alpha z_3^2 + z_4^2) = \eta(x_5^2 + x_6^2), \tag{5}$$

from (3). It follows that $x_4^2+x_6^2$ is divisible by η. Let the constant term with respect to η of x_5 be μ_5, that of x_6 be μ_6. Then obviously $\mu_5^2+\mu_6^2=0$, where μ_5 and μ_6 are polynomials in ξ with rational coefficients. If $\mu_5 \neq 0$, then $-1=(\mu_6\mu_5^{-1})^2$, which is impossible as we have shown, since -1 is not a rational square. Hence $\mu_5=\mu_6=0$ and each of the quantities x_5, x_6 is divisible by η. But then $x_1, \cdots, x_6$ are all divisible by η, a contradiction of our hypothesis. Hence we have proved that (3) is not a null form, *the direct product of our two generalized quaternion algebras*

$$B = (1, u, s, us), \quad us = -su, \quad u^2 = \tau, \quad s^2 = \sigma,$$
$$C = (1, j, t, jt), \quad tj = -jt, \quad j^2 = \gamma, \quad t^2 = \rho,$$

is a division algebra.

We now take $\alpha=\tau=1+\Delta^2$, where Δ is a rational number so chosen that $1+\Delta^2$ is not a rational square. For example Δ may be taken to be unity. Let

$$\beta_1 = \xi, \quad \gamma_0 = -\beta_1^2, \quad \nu = -\eta(2\alpha\xi^2)^{-1}.$$

The author has shown that the equation

$$\phi(\omega) \equiv \omega^4 + 2\nu(1+\Delta^2)\omega^2 + \nu^2\Delta^2(1+\Delta^2) = 0 \tag{6}$$

is a cyclic quartic over F for every $\nu \neq 0$ in F and $\tau=1+\Delta^2$ not the square of any quantity of F. Our above choices of ν and τ evidently insure that these requirements are satisfied. I have also shown (loc. cit.) that if we define the cyclic algebra with the basis

$$x^r y^s, \qquad (r, s = 0, 1, 2, 3),$$

such that

$$\phi(x) = 0, \quad y^r x = \theta^r(x) y^r, \quad y^4 = \gamma_0, \qquad (r = 0, 1, \cdots),$$

then, if $\gamma_0 = -\beta_1^2$, by Theorem 4 the algebra is expressible as the direct product of two generalized quaternion algebras B and C with

$$\tau = 1 + \Delta^2,\ \sigma = 2\beta_1\gamma_0,\ \rho = 2\nu\beta_1\tau(-\beta_1),\ \gamma = \gamma_0 = -\beta_1^2.$$

But $2\beta_1\gamma_0 = -2\xi^2$, $\rho = \eta$ and we may evidently choose a new basis with $\gamma = \gamma_0$ replaced by -1. Hence $B \times C$ is a division algebra. Moreover $\gamma_0 = -\beta_1^2$, $\gamma_0^2 = \beta_1^4$ is the norm of the scalar β_1 in $F(x)$. *For this cyclic algebra the Wedderburn norm condition is not satisfied and yet the algebra is a division algebra.*

THE UNIVERSITY OF CHICAGO

AN ACKNOWLEDGMENT

BY G. W. STARCHER

Since the publication of my article entitled *A note on geometrical factorial series* in the June issue of this Bulletin,* my attention has been called to the fact that most of the results of that paper are not new and have been previously given by F. Ryde.† Ryde considers a series which is essentially the same as (1) and calls it a "geometric factorial series." The term *geometric factorial series* was suggested to me by the fact that Cauchy had called the denominators of the terms in the series (1) *geometric factorials.*‡ Later *geometric* was changed to *geometrical.*

The conclusions in §2 are given by Ryde (page 6) and the comparison of (1) and (2) was evidently known to him. The theorem in §3 is proved in essentially the same way by Ryde (pages 6–8), and on page 8 a more general expansion is given. On pages 11–13 he employs the multiplication of such series, and he probably had formulas for the coefficients in such a product.

OHIO UNIVERSITY

---

* This Bulletin, vol. 37 (1931), pp. 455–463.

† *A contribution to the theory of linear homogeneous geometric difference equations (q-difference equations)*, Lund dissertation, published by Lindstedts Univ. Bokhandel (1921).

‡ Encyclopédie des Sciences Mathématiques, vol. I, 4, p. 279.

ON A COVARIANT DIFFERENTIATION PROCESS*

BY H. V. CRAIG

1. *Introduction.* If the components of a tensor T^{α}_{β} are point functions, a tensor of one higher covariant order may be formed by differentiation and elimination. Specifically, the equations which express the law of transformation of the tensor are differentiated once with respect to each of the coordinate variables of a given set, and certain second derivatives appearing are eliminated by means of a relationship known as the fundamental affine connection. This process is called covariant differentiation and is one of the cardinal operations of the tensor calculus.†

It is the purpose of this note to point out that a somewhat similar process exist sfor tensors whose components are functions of x, dx/dt, d^2x/dt^2.

2. *Notation.* We shall suppose that the curves involved in the following discussions are given in parametric form and shall employ primes to indicate differentiation with respect to the parameter. Partial derivatives, for the most part, will be denoted by means of subscripts. Thus, we shall write x', x'', $\left\{ {\alpha \atop \beta} \right\}_{x'^{\gamma}}$ for dx/dt, d^2x/dt^2, $\partial \left\{ {\alpha \atop \beta} \right\} / \partial x'^{\gamma}$ respectively.

3. *The Differentiation Process.* We shall illustrate this process by applying it to a mixed tensor of the second order. Accordingly, suppose that the quantities $T^{\alpha}_{\beta}(x, x', x'')$ are defined along a given regular curve and are the components of a tensor of the type indicated. The extended point transformation,

$$
\begin{aligned}
x^{\alpha} &= x^{\alpha}(y^1, y^2, \cdots, y^n), \qquad x'^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^i} y'^i, \\
x''^{\alpha} &= \frac{\partial x^{\alpha}}{\partial y^i} y''^i + \frac{\partial^2 x^{\alpha}}{\partial y^i \partial y^j} y'^i y'^j,
\end{aligned} \tag{1}
$$

* Presented to the Society, September 11, 1931.

† For an exposition of tensor analysis and covariant differentiation in particular reference may be made to Oswald Veblen, *The Invariants of Quadratic Differential Forms*, 1927, Chapters 2, 3; L. P. Eisenhart, *Riemannian Geometry*, 1926, Chapter 1.

induces the following transformation in T^{α}_{β},

$$(2) \qquad \bar{T}^{i}_{j}(y,\ y',\ y'') = T^{\alpha}_{\beta}(x, x', x'') \cdot \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial y^{j}} .$$

Differentiating (2) with respect to y'^{k} and making use of (1), we obtain

$$(3) \quad \bar{T}^{i}_{jy'k} = \left\{ T^{\alpha}_{\beta x'^{\gamma}} \cdot \frac{\partial x^{\gamma}}{\partial y^{k}} + T^{\alpha}_{\beta x''^{\gamma}} \cdot 2 \frac{\partial^{2} x^{\gamma}}{\partial y^{a} \partial y^{k}} \cdot y'^{a} \right\} \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial y^{j}} .$$

From this equation and the relationship

$$(4) \qquad \overline{\left\{ \begin{matrix} l \\ k \end{matrix} \right\}} \frac{\partial x^{\gamma}}{\partial y^{l}} = \frac{\partial^{2} x^{\gamma}}{\partial y^{k} \partial y^{a}} \cdot y'^{a} + \left\{ \begin{matrix} \gamma \\ \delta \end{matrix} \right\} \frac{\partial x^{\delta}}{\partial y^{k}},$$

$$\left(\left\{ \begin{matrix} \gamma \\ \delta \end{matrix} \right\} \equiv x'^{\beta} \Gamma^{\gamma}_{\delta\beta} + \tfrac{1}{2} x''^{\sigma} f_{\delta\beta\sigma} f^{\beta\gamma} \right),$$

which is due to J. H. Taylor,* we can eliminate the second derivatives. To facilitate this elimination we multiply (4) by $2T^{\alpha}_{\beta x''^{\gamma}} \cdot (\partial y^{i}/\partial x^{a})(\partial x^{\beta}/\partial y^{j})$ and sum, noting that by virtue of (1) $T^{\alpha}_{\beta x''^{\gamma}}$ is a tensor. The result is

$$2\bar{T}^{i}_{jy''l} \cdot \overline{\left\{ \begin{matrix} l \\ k \end{matrix} \right\}} = \left[T^{\alpha}_{\beta x''^{\gamma}} \cdot 2 \frac{\partial^{2} x^{\gamma}}{\partial y^{a} \partial y^{k}} y'^{a} + 2T^{\alpha}_{\beta x''^{\gamma}} \left\{ \begin{matrix} \gamma \\ \delta \end{matrix} \right\} \frac{\partial x^{\delta}}{\partial y^{k}} \right] \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial y^{j}} .$$

By subtracting this last equation from (3) we obtain our conclusion:

$$\bar{T}^{i}_{jy'k} - 2\bar{T}^{i}_{jy''l} \cdot \overline{\left\{ \begin{matrix} l \\ k \end{matrix} \right\}}$$

is a tensor whose covariant order is one greater than that of the original, $\bar{T}^{i}_{j}$.

Some of the obvious properties of this process are: (a), if the components of the tensor do not contain y'' the process reduces

* See J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*, Transactions of this Society, vol. 27, p. 255, equation (21). For the definition and properties of $f^{\alpha\beta}, f_{\alpha\beta\gamma}, \Gamma^{\gamma}_{\alpha\beta}$ reference may be made to pp. 247, 248, 253, ibid.

to partial differentiation; (b), the rules of ordinary calculus for the differentiation of sums and products are conserved.

Unlike ordinary covariant differentiation, however, this process is commutative with itself. To assure ourselves on this point it will suffice, perhaps, to illustrate with an arbitrary scalar $S(x, x', x'')$. On performing the evident cancellations the difference between the covariant derivatives $S_{,\beta,\gamma}$ and $S_{,\gamma,\beta}$ reduces to

$$2S_{x''}{}^{\lambda}\left[\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x'\beta} - \left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x'\gamma} + 2\left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x''\mu}\left\{\begin{matrix}\mu\\ \gamma\end{matrix}\right\} - 2\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x''\mu}\left\{\begin{matrix}\mu\\ \beta\end{matrix}\right\}\right].$$

By expanding and making use of the relation $\Gamma^{\lambda}_{\gamma\alpha} = f^{\lambda\sigma}[\gamma\alpha, \sigma]$, we obtain

$$\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x'\beta} = \Gamma^{\lambda}_{\gamma\beta} + x'^{\alpha}(f^{\lambda\delta}[\gamma\alpha, \delta])_{x'\beta} + \tfrac{1}{2}x''^{\alpha}(f_{\alpha\delta\gamma}f^{\delta\lambda})_{x'\beta},$$

$$-\left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x'\gamma} = -\Gamma^{\lambda}_{\gamma\beta} - x'^{\alpha}(f^{\lambda\delta}[\beta\alpha, \delta])_{x'\gamma} - \tfrac{1}{2}x''^{\alpha}(f_{\alpha\delta\beta}f^{\delta\lambda})_{x'\gamma},$$

$$2\left\{\begin{matrix}\lambda\\ \beta\end{matrix}\right\}_{x''\mu}\left\{\begin{matrix}\mu\\ \gamma\end{matrix}\right\} = f_{\beta\delta\mu}f^{\delta\lambda}[x'^{\alpha}f^{\mu\sigma}[\gamma\alpha, \sigma] + \tfrac{1}{2}x''^{\alpha}f_{\gamma\rho\alpha}f^{\rho\mu}],$$

$$-2\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}_{x''\mu}\left\{\begin{matrix}\mu\\ \beta\end{matrix}\right\} = -f_{\gamma\delta\mu}f^{\delta\lambda}x'^{\alpha}f[^{\mu\sigma}[\beta\alpha, \sigma] + \tfrac{1}{2}x''^{\alpha}f_{\beta\rho\alpha}f^{\rho\mu}].$$

By virtue of the relationships* $f_{\beta\delta\mu} = f_{\beta\delta x'^{\mu}}$, $f_{\beta\lambda}f^{\delta\lambda} = \delta^{\gamma}_{\beta}$ we may write $f_{\beta\delta\mu}f^{\delta\lambda} = -f_{\delta\mu}f^{\delta\lambda}_{x'\beta}$, and thus an alternative form for the right member of the third equation of the set above is

$$-x'^{\alpha}f^{\delta\lambda}_{x'\beta}[\gamma\alpha, \delta] - \tfrac{1}{2}x''^{\alpha}f_{\gamma\delta\alpha}f^{\delta\lambda}_{x'\beta}.$$

Finally, it follows from the definition† of $[\gamma\alpha, \delta]$ and the

* See J. H. Taylor, loc. cit., p. 248, line 28, and p. 249, equation (7).

† J. H. Taylor, loc. cit., equation (6), p. 248.

identity* $x'^{\alpha}f_{\alpha\beta\gamma}=0$ that $[\gamma\alpha,\ \delta]_{x'\beta}-[\beta\alpha,\ \delta]_{x'\gamma}=0$ and hence the difference in question vanishes.

A moment's consideration of the m times extended point transformation will enable us to extend the process to tensors whose components involve derivatives of any order. Thus, if the components $T^{\alpha\cdots}_{\beta\cdots}$ of a given tensor are functions of $x, x', x'', \cdots, x^{(m)}$, then the quantities

$$T^{\alpha\cdots}_{\beta\cdots x^{(m-1)}\gamma}-m\cdot T^{\alpha\cdots}_{\beta\cdots x^{(m)}\lambda}\cdot\left\{\begin{matrix}\lambda\\ \gamma\end{matrix}\right\}$$

are the components of a tensor. The commutative property established above holds in this case also.

THE UNIVERSITY OF TEXAS

ON THE DIVISIBILITY OF LOCALLY CONNECTED SPACES†

BY G. T. WHYBURN

1. *Introduction.* In this paper a property of connected, locally connected, separable metric spaces will be established which is stronger than that of *divisibility* in the sense of W. A. Wilson.‡ In order to distinguish our property from that of Wilson we shall use the term *strong divisibility*.

A space M will be said to be *strongly divisible* if for every pair of mutually exclusive closed and connected subsets A and B of M there exists a decomposition of M into three mutually exclusive sets R, F, and G, where R and G are connected and open and contain A and B, respectively, and where F is the common boundary of R and G, that is, $F=F(R)=F(G)$.

2. THEOREM. *Every connected, locally connected, separable, metric space M is strongly divisible.*

---

* J. H. Taylor, loc. cit., equation (14), p. 253.

† Presented to the Society, December 31, 1930.

‡ According to Wilson, a space M is *divisible* if for every pair of mutually exclusive subcontinua A and B of M there exists a decomposition of M into two continua P and Q such that $P\cdot B=Q\cdot A=0$. See this Bulletin, vol. 36 (1930), p. 85. Wilson's theorem that every connected, locally connected, separable metric space is divisible is obviously an immediate corollary to our theorem below in §2.

PROOF.* Let A and B be any two mutually exclusive closed and connected subsets of M. We may suppose the metric so defined in M that $\rho(A, B) = d > 0$, or in other words, M is homeomorphic with a space M' in which this is true.† For each positive number $r < d$, let $S_r(A)$ and $V_r(A)$, respectively, denote the set of all points whose distances from A are equal to and less than r. Let $U_r(A)$ denote the component of $V_r(A)$ containing A, let $G_r(B)$ be the component of $M - \overline{U_r(A)}$ containing B, let $X_r = F[G_r(B)]$, and let $R_r(A)$ be the component of $M - X_r$ containing A. Then clearly we have

$$(1) \qquad X_r = F[G_r(B)] = F[R_r(A)] \subset F[U_r(A)] \subset S_r(A).$$

Now if $r_1 < r_2 < d$, we have

$$(2) \qquad G_{r_2}(B) + X_{r_2} = \overline{G_{r_2}(B)} \subset G_{r_1}(B),$$

and therefore

$$(3) \qquad \overline{R_{r_1}(A)} \subset R_{r_2}(A), \text{ for } R_{r_1}(A) \cdot X_{r_2} = 0 \text{ by (ii)}.$$

Now let E be the set of all positive numbers $e < d$ such that there exists at least one component D_e of $M - X_e$ which is distinct from $R_e(A)$ and from $G_e(B)$. Then if e_1 and e_2 are numbers of E and $e_1 < e_2$, it follows by (2) and (3) that $D_{e_2} \subset G_{e_1}(B)$, because $F(D_{e_2}) \subset X_{e_2}$. Hence $D_{e_1} \cdot D_{e_2} = 0$. Therefore E must be countable, for the sets D_e are open. It follows that there exists a positive number $f < d$ which does not belong to E and hence such that

$$R_f(A) + G_f(B) + X_f = M,$$

and thus we have a strong division of M between A and B.

COROLLARY. *If the space M as in our theorem is locally compact, then each non-cut point p of M is contained in an arbitrarily*

* Considerable similarity will be noted between this proof and a number of proofs for closely related theorems recently published by the author. See, for example, my paper *Non-separated cuttings of connected point sets*, Transactions of this Society, vol. 33 (1931).

† In view of the theorem of Tychonoff, Mathematische Annalen, vol. 95, pp. 139–142, that our space M is normal, a glance at Urysohn's proof for the existence of a metric in normal, perfectly separable, Hausdorff spaces is sufficient to verify this statement. See P. Urysohn, *Zum Metrisationsproblem*, Mathematische Annalen, vol. 94, pp. 309–315 .

small connected neighborhood R whose exterior G is connected and whose boundary F is the boundary also of G.

For there exists* an arbitrarily small neighborhood of p whose complement B is connected. Then setting $p=A$ and applying our theorem we have the desired result.

3. *Conclusion.* In conclusion we mention some consequences of our theorem and corollary.

In the first place, it is to be noted that, by virtue of our corollary, we can define the term *region* in any non-compact but locally compact separable metric space S which is connected and locally connected and has no cut point in such a way that the topological character of the space is not altered and so that axioms 1–5 and 7′ of R. L. Moore's† system $\sum_2$ are satisfied. For this purpose it is only necessary to say that a *region* is a compact connected open set of points R whose exterior $S-\overline{R}$ is connected and has the same boundary as R. Thus it is seen that these axioms of Moore's set $\sum_2$ restrict the space only to the extent of making it a non-compact cyclicly connected continuous curve.

It is obvious that each point (whether a cut point or not) of a space M as in our corollary is contained in an arbitrarily small neighborhood R whose exterior is the sum of a finite number of connected sets and has the same boundary as R.

We note also the following application. Let J be any simple closed curve contained in the space M of our theorem, let a and b be any two points of J such that $M-(a+b)$ is connected, and let t_1 and t_2 be the two arcs of J from a to b. Then from our theorem it follows that there exists a decomposition of M into three sets R_1, R_2, and F, where R_1 and R_2 are connected and open and contain $t_1-(a+b)$ and $t_2-(a+b)$, respectively, and F is the common boundary of R_1 and R_2 and hence contains $a+b$. To prove this it is only necessary to note that in the space $M-(a+b)$, the sets $t_1-(a+b)$ and $t_2-(a+b)$ are closed and of course connected and mutually exclusive.

THE JOHNS HOPKINS UNIVERSITY

* See H. M. Gehman, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 431–432, and W. L. Ayres, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 139–140.

† See Transactions of this Society, vol. 17 (1916), p. 163.

QUADRATIC ADDITION THEOREMS FOR EVEN FUNCTIONS*

BY J. D. GRANT

The purpose of this paper is to show that if an even function be expansible in power series in some region about the origin, and satisfy a quadratic addition† theorem, the function is a linear fractional transformation of the Weierstrass "P" function or one of its degenerate forms.

P. D. Edwards‡ has shown that addition theorems are symmetric in $U=F(x)$ and $V=F(y)$. He has also shown the most general form of an addition theorem linear in $W=F(x+y)$. In this paper, equations quadratic in W are treated by means of the following theorem.

THEOREM 1. *An addition theorem for an even function is symmetric in the three variables U, V, W.*

For one may set

$$\begin{aligned} x+y=s, \quad & W=F(x+y)=F(s) && =U', \\ x=s+t, \quad & U=F(x) && =F(s+t) && =W', \\ y=-t, \quad & V=F(y) && =F(-t)=F(t) && =V'. \end{aligned}$$

Thus the equation, which is known to be symmetric in U, V, is shown to be symmetric in U, W so that the theorem follows.

In the same manner one easily proves the following companion theorem for odd functions.

THEOREM 2. *An addition theorem for an odd function is symmetric in the three variables U, V, $-W$.*

---

* Presented to the Society, April 3, 1931.

† An *algebraic addition theorem* is a polynomial relation between $F(x)$, $F(y)$, and $F(x+y)$, with constant coefficients. For this paper the polynomial is irreducible. Weierstrass made addition theorems the basis for his study of elliptic functions. Osgood, *Allgemeine Funktionentheorie*, 1912, vol. 1, pp. 580–595; Forsyth, *Theory of Functions*, 1918, Chapter XIII; Hancock, *Theory of Elliptic Functions*, 1910.

‡ Abstract No. 8, this Bulletin, vol. 35 (1929), p. 453. The paper is available at the University of Indiana Library.

This theorem is illustrated by $\tan x$ and $\operatorname{ctn} x$, but the form is lost for $\sin x$ and Jacobi's $\operatorname{sn} x$, whose addition theorems are symmetric in U^2, V^2, W^2.

The equation whose solution we wish to study is the following, symmetric in the three variables U, V, W:

$$\begin{aligned}(1)\quad & a_0 + a_1(U+V+W) + a_2(UV+UW+VW) \\ & + a_3(U^2+V^2+W^2) + a_4UVW \\ & + a_5(UV^2+U^2V+UW^2+U^2W+VW^2+V^2W) \\ & + a_6(U^2V^2+U^2W^2+V^2W^2) + a_7UVW(U+V+W) \\ & + a_8UVW(UV+UW+VW) + a_9U^2V^2W^2 = 0.\end{aligned}$$

This may also be written in powers of W as

$$(2)\qquad R_1(U,V)W^2 - 2R_2(U,V)W + R_3(U,V) = 0,$$

where R_1, R_2, R_3 are of degree not more than two in U or V.

Since we are dealing with even functions, changing y to $-y$ alters only W. Hence the two roots of this quadratic are $W = F(x+y)$, $F(x-y)$, and by the ordinary theorems regarding symmetric functions of the roots one has

$$(3)\quad F(x+y) + F(x-y) = 2R_2/R_1, F(x+y)\cdot F(x-y) = R_3/R_1.$$

Thus two other interesting types of functional equations are seen to arise naturally and will be solved when R_1, R_2, R_3 are quadratics symmetric in U, V.

It has been shown* that, if the first term of the power series expansion of $F(x)$ is not a constant, there is in the functional equation a group of terms which form an equation having the first term of the series as a solution. Since first terms of the form c_nx^n, $n<-2$, or $n>2$, would require the vanishing of all the coefficients a in (1), we have to consider only three possible first terms, $c_{-2}x^{-2}$, c_0, c_2x^2. In case the first term is c_0, one writes

$$U = U_1 + c_0$$

and the first term for U_1 will be c_2x^2. In case the first term is c_2x^2, one writes

$$U = 1/U'$$

* The author in American Mathematical Monthly, vol. 37, p. 70.

and the first term for U' will be $c_{-2}x^{-2}$. The transformed equation in U_1, V_1, W_1, or U', V', W' will still be of type (1). Hence we see that any solution of an equation (1) is a linear fractional transformation of a solution of a type (1) equation, whose series expansion begins with $c_{-2}x^{-2}$, $c_{-2}\neq 0$. For this reduced case, equation (1) contains the terms of the addition theorem for x^{-2} as its terms of highest degree and they must enter with a non-zero multiplier (chosen as unity). This gives our equation (1) as

$$a_0 + a_1(U+V+W) + a_2(UV+UW+VW) + a_3(U^2+V^2+W^2) + a_4UVW + a_5(UV^2+U^2V+UW^2+U^2W+VW^2+V^2W) + U^2V^2+U^2W^2+V^2W^2-2UVW(U+V+W)=0.$$

The solution has the expansion

$$U = F(x) = \frac{1}{x^2} + c_0 + c_2x^2 + c_4x^4 + \cdots.$$

For $x=y$, the roots of (2) become

$$F(2x), \quad F(0)=\infty, \quad \text{and} \quad R_1(U,U)\equiv 0,$$

so that $a_5=a_3=0$. Equation (1) is further reduced to

$$(4)\quad a_0 + a_1(U+V+W) + a_2(UV+UW+VW) + a_4UVW + U^2V^2+U^2W^2+V^2W^2-2UVW(U+V+W)=0.$$

Equation (4) is invariant in form under the transformation $U=U'+t$ and the cubic term becomes $(a_4-12t)U'V'W'$. When $t=a_4/12$, equation (4) becomes, if we omit accents,

$$(5)\quad b_0 + b_1(U+V+W) + b_2(UV+VW+UW) + U^2V^2+U^2W^2+V^2W^2-2UVW(U+V+W)=0.$$

If we set $x=y$ in (5), substitute the power series for the solution, and after collecting terms equate to zero the coefficients of each power of x, the first four conditions on c_i are:

$$(6)\quad \begin{aligned} x^{-6}&: \quad c_0 = 0,\\ x^{-4}&: \quad 10c_2 - b_2 = 0,\\ x^{-2}&: \quad 28c_4 - b_1 = 0,\\ x^{0}&: \quad 2b_0 + 21b_2c_2 - 90c_2^3 - 510c_6 = 0. \end{aligned}$$

Again if we set $y=2x$ in (5) and make the same computation, there results:

$$
\begin{aligned}
&x^{-6}: && c_0 = 0,\\
&x^{-4}: && 10c_2 - b_2 = 0,\\
&x^{-2}: && 28c_4 - b_1 = 0,\\
&x^{0}: && 18b_0 + 289b_2c_2 - 1210c_2^2 - 6390c_6 = 0.
\end{aligned} \tag{7}
$$

The conditions (6) and (7) require for consistency that $4b_0=b_2^2$. To complete the proof one notices that if $b_2=g_2/2$, $b_1=g_3$, $b_0=b_2^2/4=g_2^2/16$, the equation (1) has been reduced by means of linear fractional transformations to the well known addition theorem* for $P(x; g_2, g_3)$. In this connection the degenerate forms of $P(x; g_2, g_3)$ should be taken as $(2/3+\text{ctn}^2x)$ and x^{-2}, as these have $c_0=0$.

It will be seen from a study of the addition theorem for $P(x; g_2, g_3)$ that every second-degree addition theorem having an even solution may be written

$$W = \frac{R_2(UV) \pm kU'V'}{R_1(UV)},$$

where U' and V' are the derivatives of U and V with respect to x and y respectively.

JAMES MILLIKIN UNIVERSITY

* Since the addition theorems give rise to the differential equation by a direct computation, (see Forsyth, p. 357), it is one of the many ways of defining the P function.

TRANSFORMATIONS OF DOUBLE SEQUENCES WITH APPLICATION TO CESÀRO SUMMABILITY OF DOUBLE SERIES*

BY C. R. ADAMS

1. *Introduction.* Of the definitions of convergence commonly employed for double series, only that due to Pringsheim† permits a series to converge conditionally. Therefore, in spite of any disadvantages which it may possess, this definition is better adapted than others to the study of many problems in double sequences and series.

Chief among the reasons why the theory of double sequences, under the Pringsheim definition of convergence, presents difficulties not encountered in the theory of simple sequences is the fact that a double sequence $\{x_{ij}\}$ may converge without x_{ij} being a bounded function of i and j. Thus it is not surprising that many authors in dealing with the convergence of double sequences should have restricted themselves to the class of bounded sequences, or in dealing with the summability of double series, to the class of series for which the function whose limit is the sum of the series is a bounded function of i and j. Without such a restriction, peculiar things may sometimes happen; for example, a double power series may converge with partial sum S_{ij} unbounded at a place exterior to its associated circles of convergence.

Nevertheless there are problems in the theory of double sequences and series where this restriction of boundedness as it has been applied is considerably more stringent than need be. It is the purpose of the present paper to prove a general theorem concerning the regularity of transformations of double sequences when the original double sequence is not necessarily bounded, and to apply this theorem to the question of consistency of Cesàro summability of double series. The theorem immediately suggests numerous extensions and generalizations: extensions of results already known for certain classes of double sequences

* Presented to the Society, September 8, 1931.

† See Pringsheim, Encyklopädie der mathematischen Wissenschaften, vol. I, A3, pp. 97–98.

and series to new and broader classes; generalizations of well known results concerning simple sequences to certain classes of double sequences. Some of these problems we hope to consider in a later paper.

In recent years a considerable number of writers have given attention to the question of transformations of multiple sequences or series and related topics; these include especially Bromwich and Hardy,* C. N. Moore,† Smail,‡ Kojima,§ Eversull,‖ Robison,¶ Merriman,** and Mears.†† So far as we are aware, the only ones to obtain results of regularity or consistency without imposing the condition of boundedness on the original sequence or series are Kojima and Miss Mears.

It should be emphasized that one may at once extend the following work, without meeting additional difficulties, to multiple sequences or series of any order of multiplicity.

2. *Transformations of Double Sequences.* Let $\{x_{mn}\}$ be a double sequence and $\|a_{mnkl}\|$, $(m, n, k, l=1, 2, 3, \cdots)$, a four-dimensional matrix of numbers, real or complex, with $a_{mnkl}=0$, for $k>m$ or $l>n$ or both. Then the transformation

$$y_{mn} = \sum_{k=1,l=1}^{m,n} a_{mnkl}x_{kl},$$

which we term *a double A-transformation*, defines a new double

* Bromwich and Hardy, *Some extensions to multiple series of Abel's theorem on the continuity of power series*, Proceedings of the London Mathematical Society, (2), vol. 2 (1904), pp. 161–189.

† C. N. Moore, *On convergence factors in double series and the double Fourier's series*, Transactions of this Society, vol. 14 (1913), pp. 73–104; *On convergence factors in multiple series*, ibid., vol. 29 (1927), pp. 227–238.

‡ Smail, *Summability of double series*, Annals of Mathematics, (2), vol. 21 (1920), pp. 221–223.

§ Kojima, *On the theory of double sequences*, Tôhoku Mathematical Journal, vol. 21 (1922), pp. 3–14.

‖ Eversull, *On convergence factors in triple series and the triple Fourier's series*, Annals of Mathematics, (2), vol. 24 (1922–23), pp. 141–166.

¶ Robison, *Divergent double sequences and series*, Transactions of this Society, vol. 28 (1926), pp. 50–73.

** Merriman, *Concerning the summability of double series of a certain type*, Annals of Mathematics, (2), vol. 28 (1927), pp. 515–533; *A set of necessary and sufficient conditions for the Cesàro summability of double series*, ibid., (2), vol. 29 (1928), pp. 343–354.

†† Mears, *Riesz summability for double series*, Transactions of this Society, vol. 30 (1928), pp. 686–709.

sequence $\{y_{mn}\}$. *If the transformation has the property that it carries over every convergent sequence* $\{x_{mn}\}$ ***of a certain class*** *into a sequence* $\{y_{mn}\}$ *convergent to the same limit, it is said to be* ***a regular transformation for this class of sequences.*** We are interested in determining sufficient conditions for regularity of a double A-transformation for sequences of a certain class.*

It may be recalled that a simple A-transformation, or A-transformation for simple sequences, is defined by means of a matrix

$$\|a_{mk}\| = \begin{Vmatrix} a_{11} & 0 & 0 & 0 \cdots \\ a_{21} & a_{22} & 0 & 0 \cdots \\ a_{31} & a_{32} & a_{33} & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots \end{Vmatrix},$$

and that the Silverman-Toeplitz necessary and sufficient conditions for regularity of this transformation are

$$\lim_{m\to\infty} a_{mk} = 0, \qquad (k = 1, 2, 3, \cdots),$$

$$\lim_{m\to\infty} \sum_{k=1}^{m} a_{mk} = 1, \qquad \sum_{k=1}^{m} |a_{mk}| < K, \qquad (m = 1, 2, 3, \cdots). \tag{1}$$

A double A-transformation will be said to be the "product" of two simple A-transformations, A' and A'', whenever we have $a_{mnkl} = a'_{mk}a''_{nl}$, $(m, n, k, l = 1, 2, 3, \cdots)$; we shall then write $A = A' \odot A''$. On account of the trend of development taken by the theory of summability of double series, we are particularly interested in double A-transformations of this character. Concerning them we now establish the following theorem.

THEOREM 1. *Let A' and A'' be any two regular simple A-transformations. Then the double A-transformation $A' \odot A''$ is regular for the class of double sequences of which each row is transformed by a'', and each column by A', into a bounded sequence.*

PROOF. It follows at once from the definition that every double A-transformation is distributive. Moreover, since A' and A'' are regular and the second of conditions (1) is consequently satisfied by each, it is seen that any constant sequence, $\{x_{mn}\} = \{c\}$,

* Conditions both necessary and sufficient for regularity of a double A-transformation for the class of *bounded* sequences have been found by Robison, loc. cit., p. 53.

is transformed by the double A-transformation $A' \odot A''$ into a sequence converging to c. Hence we may, without loss of generality, confine ourselves to proving that if $\{x_{mn}\}$ converges to *zero*, $\{y_{mn}\}$ also converges to zero; that is, that given any $\epsilon > 0$, there exist positive numbers M, N such that

$$(2) \qquad |y_{mn}| = \left| \sum_{k=1,l=1}^{m,n} a_{mnkl} x_{kl} \right| < \epsilon,$$

for $m > M$, $n > N$. Let any $\epsilon > 0$ be assigned. A' and A'' being regular, a'_{mk} and a''_{nl} satisfy the conditions (1), where one K may serve for both transformations. And since $\{x_{mn}\}$ converges to zero, there exist positive integers p, q such that we have

$$(3) \qquad |x_{mn}| < \epsilon/(4K^2),$$

for $m > p$, $n > q$. Let p, q be held fast; henceforth we restrict ourselves to values of $m > p$ and of $n > q$. The left member of (2) is then at most equal to

$$(4) \qquad \left| \sum_{k=1,l=1}^{p,q} \right| + \left| \sum_{k=p+1,l=q+1}^{m,n} \right| + \left| \sum_{k=p+1,l=1}^{m,q} \right| + \left| \sum_{k=1,l=q+1}^{p,n} \right|,$$

in which the expression operated on by $\sum$ is in each case $a_{mnkl} x_{kl}$. We consider the four terms of (4) seriatim.

Let D (>0) be at least equal to the largest of the quantities

$$|x_{kl}|, \quad (k = 1, 2, \cdots, p; l = 1, 2, \cdots, q).$$

By (1) we have

$$\lim_{m,n \to \infty} a_{mnkl} = \lim_{m \to \infty} a'_{mk} \lim_{n \to \infty} a''_{nl} = 0.$$

Hence there exist numbers M_1, N_1 such that we have, for $m > M_1$, $n > N_1$,

$$|a_{mnkl}| < \epsilon/(4pqD), \qquad (k = 1, 2, \cdots, p; l = 1, 2, \cdots, q);$$

and the first term of (4) is at most equal to

$$\sum_{k=1,l=1}^{p,q} \epsilon |x_{kl}| / (4pqD) \leqq \epsilon/4$$

for $m > M_1$, $n > N_1$. By (3) and (1) the second term of (4) is less than or equal to

$$\sum_{k=p+1,\,l=q+1}^{m,n} |a_{mnkl}|\,\epsilon/(4K^2) = \frac{\epsilon}{4K^2}\sum_{k=p+1}^{m}|a'_{mk}|\sum_{l=q+1}^{n}|a''_{nl}| < \epsilon/4.$$

The third term of (4) is less than or equal to

$$\begin{aligned}\sum_{l=1}^{q}\left|\sum_{k=p+1}^{m}a_{mnkl}x_{kl}\right| &= \sum_{l=1}^{q}|a''_{nl}|\cdot\left|\sum_{k=p+1}^{m}a'_{mk}x_{kl}\right| \\ &\leqq \sum_{l=1}^{q}|a''_{nl}|\cdot\left[\left|\sum_{k=1}^{m}a'_{mk}x_{kl}\right| + \left|\sum_{k=1}^{p}a'_{mk}x_{kl}\right|\right] \\ &\leqq \sum_{l=1}^{q}|a''_{nl}|\cdot\left[B_l + D\sum_{k=1}^{p}|a'_{mk}|\right],\end{aligned}$$

where B_l (>0) denotes a bound for the A'-transform of the lth column of $\{x_{mn}\}$. Let B be at least equal to the largest of the bounds B_l, $(l=1, 2, \cdots, q)$; then by (1) we infer the existence of numbers M_2, N_2 such that we have

$$|a'_{mk}| < B/(pD), \qquad (\text{for } m > M_2;\ k = 1, 2, \cdots, p);$$
$$|a''_{nl}| < \epsilon/(8qB), \qquad (\text{for } n > N_2;\ l = 1, 2, \cdots, q).$$

Thus the third term of (4) is less than $\epsilon/4$ for $m>M_2$, $n>N_2$. Similarly there exist numbers M_3, N_3 such that the fourth term of (4) is less than $\epsilon/4$ for $m>M_3$, $n>N_3$.

Hence if M be taken as the largest of M_1, M_2, M_3, and N as the largest of N_1, N_2, N_3, the inequality (2) will be satisfied. This completes the proof.

If we do not restrict ourselves to double A-transformations which are the products of simple A-transformations, the above proof can easily be modified to yield a second theorem, of which Theorem 1 is a particular case. For simplicity of statement let us first recall Robison's generalization of the Silverman-Toeplitz theorem.*

Necessary and sufficient conditions that a double A-transformation be regular **for the class of bounded sequences** *are*

$$\lim_{m,n\to\infty} a_{mnkl} = 0, \qquad (\textit{for each } k \textit{ and } l);$$

and

* Robison, loc. cit., p. 53.

$$\lim_{m,n\to\infty} \sum_{k=1,l=1}^{m,n} a_{mnkl} = 1;$$

$$\lim_{m,n\to\infty} \sum_{k=1}^{m} |a_{mnkl}| = 0, \qquad (\textit{for each } l);$$

$$\lim_{m,n\to\infty} \sum_{l=1}^{n} |a_{mnkl}| = 0, \qquad (\textit{for each } k);$$

$$\sum_{k=1,l=1}^{m,n} |a_{mnkl}| < K, \quad (K = \textit{constant}).$$

We then have the following theorem.

THEOREM 2. *If a double A-transformation is regular for the class of bounded sequences, it is also regular for the class of sequences* $\{x_{mn}\}$ *satisfying the following conditions*: (a) *each simple sequence* $\{x_{ml}\}$ (l *fixed*) *is carried by the transformation*

$$u_m^{nl} = \sum_{k=1}^{m} a_{mnkl}x_{kl}, \qquad (n \textit{ fixed}),$$

into a sequence $\{u_m^{nk}\}$ *whose elements are bounded by a constant* B_{nl} *such that, for each* l, $\lim_{n\to\infty} B_{nl} = 0$; (b) *each simple sequence,* $\{x_{kn}\}$ (k *fixed*) *is carried by the transformation*

$$v_n^{mk} = \sum_{l=1}^{n} a_{mnkl}x_{kl}, \qquad (m \textit{ fixed}),$$

into a sequence $\{v_n^{mk}\}$ *whose elements are bounded by a constant* C_{mk} *such that, for each* k, $\lim_{m\to\infty} C_{mk} = 0$.

3. *Application to Cesàro Summability of Double Series.* Moore* has defined summability (C, r) of a double series

$$\sum_{i=1,j=1}^{\infty,\infty} u_{ij} \tag{5}$$

in the following natural manner. Let us employ the notation

$$S_{mn} = \sum_{i=1,j=1}^{m,n} u_{ij}, \qquad S_{mn}^{(r)} = \sum_{i=1,j=1}^{m,n} A_{m-i+1}^{(r-1)} \cdot A_{n-j+1}^{(r-1)} S_{ij},$$

$$A_l^{(r)} = \frac{\Gamma(r+l)}{\Gamma(r+1)\Gamma(l)}, \qquad A_{mn}^{(r)} = A_m^{(r)} \cdot A_n^{(r)}.$$

* Moore, loc. cit., Transactions of this Society, 1913, p. 74.

Then if as m and n become infinite simultaneously but independently the limit of $S_{mn}^{(r)}/A_{mn}^{(r)}$ exists, the series (5) is said to be summable (C, r). He gave a proof, valid for series of real terms only, of the following lemma.

MOORE'S CONSISTENCY LEMMA. *If the series* (5) *is summable* (C, r), *where* r *is zero or any positive integer, and if furthermore we have*

$$|S_{mn}^{(r)}/A_{mn}^{(r)}| < C, \quad (m, n = 1, 2, 3, \cdots), \tag{6}$$

where C *is a constant, then the series is summable* $(C, r+1)$ *to the same sum and in addition we have*

$$|S_{mn}^{(r+1)}/A_{mn}^{(r+1)}| < C, \ (m, n = 1, 2, 3, \cdots).$$

Moore expressed doubt as to whether the condition (6) were necessary. We now propose to show that it is not necessary by applying the results of §2, and to obtain theorems valid for series of complex as well as of real terms. To this end let us recall Moore's equations (26) and (27):

$$S_{mn}^{(r+1)} = \sum_{i=1,j=1}^{m,n} S_{ij}^{(r)}, \quad A_{mn}^{(r+1)} = \sum_{i=1,j=1}^{m,n} A_{ij}^{(r)},$$

$$S_{mn}^{(r)} = \Delta_{11} S_{m-1,n-1}^{(r+1)}, \quad A_{mn}^{(r)} = \Delta_{11} A_{m-1,n-1}^{(r+1)}, \tag{7}$$

where, in general,

$$\Delta_{11} C_{m-1,n-1} = C_{m,n} - C_{m,n-1} - C_{m-1,n} + C_{m-1,n-1}.$$

We are interested in determining conditions under which, when the sequence

$$\{S_{mn}^{(r)}/A_{mn}^{(r)}\} = \{\Delta_{11} S_{mn}^{(r+1)}/\Delta_{11} A_{mn}^{(r+1)}\} \tag{8}$$

converges, the sequence

$$\{S_{mn}^{(r+1)}/A_{mn}^{(r+1)}\} \tag{9}$$

will converge to the same limit. By means of the relations (7) it is easily shown that the sequence (8) is carried over into the sequence (9) by the transformation whose general term, a_{mnkl} in the notation of §1, is

$$a_{mnkl} = \frac{A_{kl}^{(r)}}{\sum_{i=1,j=1}^{m,n} A_{ij}^{(r)}} = \frac{A_k^{(r)} \cdot A_l^{(r)}}{\sum_{i=1}^{m} A_i^{(r)} \sum_{j=1}^{n} A_j^{(r)}} = \frac{A_k^{(r)}}{A_m^{(r+1)}} \cdot \frac{A_l^{(r)}}{A_n^{(r+1)}}.$$

Each of these factors is the general term of the simple A-transformation by which the (C, r)-transform of a simple sequence is carried over into the $(C, r+1)$-transform of that sequence; this transformation is at once seen to be regular. Hence by Theorem 1, taking into account the fact that S_{ij} is the sum of partial sums for i rows or j columns of (5), we obtain the following consistency theorem.

THEOREM 3. *If the series* (5) *is summable* (C, r), *where* r *is zero or any positive integer, and if each row and column of* (5) *is finite* $(C, r+1)$, *then the series is summable* $(C, r+1)$ *to the same sum.*

In a similar manner we may establish the following more general consistency theorem.

THEOREM 4. *If the series* (5) *is summable* (C, r), *where* r *is zero or any positive integer, and if each row and column of* (5) *is finite* $(C, r+p)$ *for any positive integer* p, *then the series is summable* $(C, r+p)$ *to the same sum.*

Since a simple series finite $(C, r+p)$ is finite $(C, r+p+p')$ for any positive integer p', it follows that under the conditions stated in this theorem, the series (5) is also summable $(C, r+p+p')$ to the same sum.

If in defining Cesàro summability of (5), Cesàro factors of different orders, say r and s, are associated with rows and columns,* we may by a similar discussion obtain the following result.

THEOREM 5. *If the series* (5) *is summable* (C, r, s), *where each of the integers* r, s *is* $\geqq 0$, *and if each row of* (5) *is finite* $(C, s+q)$ *and each column finite* $(C, r+p)$, *where each of the integers* p, q *is* $\geqq 0$, *then the series is summable* $(C, r+p, s+q)$.

BROWN UNIVERSITY

* See Merriman, loc. cit., Annals of Mathematics, 1927, p. 525.

USEFUL FUNCTIONS ASSOCIATED WITH RATIONAL CUBIC CURVES*

BY CLIFFORD BELL

1. *Introduction.* The usual method of determining the Plücker numbers for plane curves is generally laborious. However, in certain cases, by the use of special theorems, the numbers may be determined at once. Thus, the curve $x_1 : x_2 : x_3 = f(t) : \phi(t) : \psi(t)$, where f, ϕ, ψ are polynomials of degree 3 in t and the coefficients of t^2 in each are zero, has one cusp.† Also any cusp or node at a vertex of the triangle of reference is easily recognized by the form of the equation. The purpose of this paper is to derive some of the properties of two functions that are useful in determining whether the rational cubic is nodal or cuspidal.

2. *The Cubic Circumscribed about the Triangle of Reference.* The parametric equations of the cubic are

$$\rho x_i = (\lambda - \lambda_i)(\lambda - \lambda_{i+2})(\lambda - s_i), \qquad (i = 1, 2, 3), \tag{1}$$

where $\lambda_i, \lambda_{i+2}, s_i$ are the points of intersection of the cubic with the side x_i of the triangle, λ_i and λ_{i+2} being vertices. Let A represent the function

$$(\lambda_3 - s_1)(\lambda_1 - s_2)(\lambda_2 - s_3) - (\lambda_1 - s_1)(\lambda_2 - s_2)(\lambda_3 - s_3).$$

THEOREM 1. *The cubic* (1) *has a cusp at one of the vertices of the triangle of reference or is nodal when A vanishes.*

PROOF. The class of the cubic is given by the degree of λ in the equation of a tangent line to the curve from any point (x_1, x_2, x_3), after all common factors are eliminated. The tangent line is given by the equation

$$\sum_{i=1}^{3} x_i(a_i\lambda^4 + b_i\lambda^3 + c_i\lambda^2 + d_i\lambda + e_i) = 0, \tag{2}$$

where

$$a_i = -\lambda_i + \lambda_{i+2} + s_{i+2} - s_{i+1},$$

$$e_i = \lambda_{i+1}^2(\lambda_i\lambda_{i+2}s_{i+2} + \lambda_{i+2}s_{i+1}s_{i+2} - \lambda_i s_{i+1}s_{i+2} - \lambda_i\lambda_{i+2}s_{i+1}),$$

* Presented to the Society, June 13, 1931.

† For the general theorem, see Hilton, *Plane Algebraic Curves*, 1920, p. 151.

and b_i, c_i, d_i are other functions of the same quantities. It is readily seen that not more than one of a_1, a_2, a_3 can be zero without all being zero. When $A = a_1 = 0$,

$$a_2 = 2\frac{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)}{(s_3 - 2\lambda_2 + \lambda_3)} \neq 0,$$

and

$$a_3 = 2\frac{(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}{(s_2 - 2\lambda_2 + \lambda_1)} \neq 0.$$

Also if $a_2 = 0$, then a_1 and a_3 are not zero; or if $a_3 = 0$, a_1 and a_2 are different from zero. Hence the cubic is nodal unless the coefficients of x_1, x_2, x_3 in (2) have a common factor.

Suppose the coefficients of x_1, x_2, x_3 have a common factor $\lambda + \alpha$. Make the transformation $\lambda = \lambda' - \alpha$ on λ. The equation of the tangent line to the curve from any point (x_1, x_2, x_3) is given by (2) with primes on all the letters in the coefficients of x_1, x_2, x_3. In particular,

$$e_i = \lambda_{i+1}^{\prime 2}(\lambda_i'\lambda_{i+2}'s_{i+2}' + \lambda_{i+2}'s_{i+1}'s_{i+2}' - \lambda_i' s_{i+1}'s_{i+2}' - \lambda_i'\lambda_{i+2}'s_{i+1}'), \qquad (i = 1, 2, 3),$$

where $\lambda_i' = \lambda_i + \alpha$, $s_i' = s_i + \alpha$. Under this transformation it is seen that

$$A' = (\lambda_3' - s_1')(\lambda_1' - s_2')(\lambda_2' - s_3') - (\lambda_1' - s_1')(\lambda_2' - s_2')(\lambda_3' - s_3') = A.$$

As $\lambda + \alpha$ is a factor of the old coefficients, λ' is a factor of the new ones. Therefore e_1', e_2', e_3' are all zero. This happens when and only when $\lambda_i' = s_i' = s_{i+1}' = 0$, which gives a cusp at a vertex.

It should be noted in the application of this theorem that a cubic with a cusp at a vertex of the triangle of reference is easily recognized, for a common square factor appears in two of the three parametric equations of the curve. Then, when A vanishes, the cubic is nodal if it is not of the above type.

THEOREM 2. *If the cubic* (1) *is cuspidal, A is zero for those cubics for which the cusp falls at one of the vertices of the triangle of reference, and different from zero for all others.*

PROOF. Either a_1, a_2, a_3 vanish or the coefficients of x_1, x_2, x_3 have a common factor for a cuspidal cubic. If $a_1=a_2=a_3=0$, $A=2(s_1-s_2)(s_3-s_2)(s_3-s_1)\neq 0$. If the coefficients of x_1, x_2, x_3 have a common factor, $\lambda+\alpha$, make the transformation $\lambda=\lambda'-\alpha$. The resulting constant terms, e_1', e_2', e_3', are all zero. It is easily shown that $A'=A\neq 0$ unless $\lambda_1'=s_1'=s_2'=0$, $\lambda_2'=s_2'=s_3'=0$ or $\lambda_3'=s_1'=s_3'=0$, in which cases a cusp falls at a vertex of the triangle of reference and $A=0$. For these cases $\lambda_1=s_1=s_2=\alpha$, $\lambda_2=s_2=s_3=\alpha$, or $\lambda_3=s_1=s_3=\alpha$.

THEOREM 3. *A triangle of reference, inscribed to any nodal cubic, can always be chosen such that A vanishes.*

PROOF. By a suitable choice of coordinates any crunodal cubic can be put in the form* $x_1:x_2:x_3=\lambda(\lambda^2-1):(\lambda^2-1):\lambda^3$ and any acnodal cubic in the form $x_1:x_2:x_3=\lambda(\lambda^2+1):(\lambda^2+1):\lambda^3$. A transformation

$$\begin{aligned}\rho y_1 &= (39x_1 - 14x_2 - 24x_3)/15,\\ \rho y_2 &= (-17x_1 - 30x_2 + 24x_3)/7,\\ \rho y_3 &= (34x_1 - 21x_2 - 24x_3)/10\end{aligned}$$

puts the crunodal cubic in the form (1), where λ_1, λ_2, λ_3, s_1, s_2, s_3 are, respectively, 2, 3, $-7/5$, $1/3$, $-5/7$, $1/2$ and for which $A=0$. Likewise a transformation can be found for which the acnodal cubic goes into the form (1) with A vanishing.

3. *The Cubic Inscribed to the Triangle of Reference.* The parametric equations of the cubic are

$$\rho x_i = (t-t_i)^2(t-r_i), \qquad (i=1, 2, 3), \tag{3}$$

where t_i and r_i are the parameters of the points of contact and intersection, respectively, of the cubic with $x_i=0$. Let the function

$$(t_3-r_1)(t_1-r_2)(t_2-r_3)-(t_2-r_1)(t_3-r_2)(t_1-r_3)$$

be represented by B.

THEOREM 4. *The cubic given by equations* (3) *has a cusp on one of the lines* $x_1=0$, $(i=1, 2, 3)$, *or is nodal when B vanishes.*

* Durège, Mathematische Annalen, vol. 1 (1869), pp. 513–515.

The proofs for this theorem and the next two are very similar to the proofs of the first three and hence will not be given. The cases where a cusp falls on the lines $x_i=0$ may easily be recognized. Thus if any two of t_1, t_2, t_3 are equal, a cusp falls at a vertex of the triangle of reference, and if the corresponding pair from r_1, r_2, r_3 are equal, B vanishes. A cusp on the lines $x_1=0$ but not at a vertex may be found by noting that t_1, t_2, t_3 are the only possible parameters for such a cusp and that the tangent line through it is indeterminate. In this case also B may vanish.

THEOREM 5. *If the cubic* (3) *is cuspidal, B may or may not be zero when the cusp is on one of the lines $x_i=0$, $(i=1, 2, 3)$, and B is different from zero for all other cases.*

THEOREM 6. *A triangle of reference, circumscribed to any nodal cubic, can always be chosen such that B vanishes.*

THE UNIVERSITY OF CALIFORNIA AT LOS ANGELES

AN APPLICATION OF METRIC GEOMETRY TO DETERMINANTS*

BY L. M. BLUMENTHAL

1. *Introduction.* A paper presented to the Accademia dei Lincei by B. Segre† is devoted to the following theorem, announced by H. W. Richmond:‡

If in a non-vanishing, symmetric determinant of order six, the six elements in the principal diagonal are all zero, and the complementary minors of five of these elements are also zero, then the complementary minor of the remaining element must be zero.

Segre states that the analogous theorem for determinants of the second§ and the fourth orders may be immediately verified, and the object of his investigation is to ascertain if analogous theorems are valid for determinants of other orders. He shows

* Presented to the Society, September 9, 1931.

† *Intorno ad una proprietà dei determinanti simmetrici del 6° ordine*, Atti dei Lincei, (6), vol. 2 (1925), p. 539.

‡ *On the property of a double-six of lines, and its meaning in hypergeometry*, Proceedings of the Cambridge Philosophical Society, vol. 14 (1908), p. 475. The statement of the theorem given in this paper contains no explicit hypothesis relative to the non-vanishing of the determinant.

§ The theorem is, of course, trivial for second-order determinants.

that such theorems may possibly hold for determinants of the eighth order, and that this is the only possible additional extension of Richmond's theorem.

The purpose of this note is to show by means of examples that, far from being readily verified for determinants of order 4, *the analogous theorem is not true for fourth-order* determinants. It follows immediately that the theorem announced is not true for determinants of order six, nor the analogous theorem for determinants of order eight.

The emphasis of this note, however, is laid upon a "counter-theorem"* suggested by considerations in *metric geometry* (in the sense of Karl Menger).

2. *Failure of Richmond's Theorem.* Consider the symmetric determinant

$$D_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = 1.$$

The complementary minor of the element in the first row and first column has the value 2, whereas the complementary minors of the other three elements in the principal diagonal are all zero. Hence D_4 is a fourth-order determinant satisfying the hypotheses of Richmond's theorem stated for fourth-order determinants, and for which the conclusion of the theorem does not hold.

To obtain a sixth-order determinant for which the theorem is false, it is only necessary to border the determinant D_4 obtaining

$$D_6 = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}.$$

* This theorem was suggested by Karl Menger, to whom I am indebted for valuable counsel in preparing this note.

Then $D_6 = -D_4 = -1$, and the only element in the principal diagonal that has a non-vanishing complementary minor is the element in the first row and the first column. It is clear that a determinant of order eight that violates the theorem stated for determinants of this order can be formed from D_6 in a manner similar to the way in which D_6 was formed from D_4.

3. *The Counter Theorem.* Segre shows that determinants of order 5 may be constructed for which the analogous theorem is not valid. He exhibits also a fifth-order determinant that *does* satisfy such a theorem. We shall prove the following theorem.

THEOREM. *If the symmetric determinant*

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12} & r_{13} & r_{14} \\ 1 & r_{21} & 0 & r_{23} & r_{24} \\ 1 & r_{31} & r_{32} & 0 & r_{34} \\ 1 & r_{41} & r_{42} & r_{43} & 0 \end{vmatrix}, \qquad (r_{ij} = r_{ji}),$$

with $r_{ij} > 0$, $(i, j = 1, 2, 3, 4)$, $i \neq j$, is different from zero, and the complementary minors of four of the elements in the principal diagonal vanish, then the complementary minor of the remaining element does not vanish.

Since the elements r_{ij} are all positive, we may set $r_{ij} = (ij)^2$, where we interpret $(ij)^2$ as the square of the distance between a point p_i and a point p_j of a quadruple p_1, p_2, p_3, p_4. Suppose now that the conditions of the theorem are satisfied. Then there are, *a priori*, two cases possible.

CASE A. *The four bordered complementary minors all vanish.* We show then that the fifth minor

$$\mathcal{E}(p_1, p_2, p_3, p_4) = \begin{vmatrix} 0 & (12)^2 & (13)^2 & (14)^2 \\ (21)^2 & 0 & (23)^2 & (24)^2 \\ (31)^2 & (32)^2 & 0 & (34)^2 \\ (41)^2 & (42)^2 & (43)^2 & 0 \end{vmatrix}$$

does not vanish.

Now the vanishing of a bordered minor (a bordered determi-

nant of three points) means, according to the Heronian formula,* that one of the points is between† the two others; that is, that the three points are linear. Hence for Case A all four of the triples that are selected from the four points p_1, p_2, p_3, p_4 are linear. On the other hand, the determinant D is not zero. This implies that the four points p_1, p_2, p_3, p_4 are not linear, but form a pseudo-linear quadruple.‡ Then the following relations subsist:§

$$(12) = (34) = a \qquad (23) = (14) = b, \qquad (31) = (24) = a + b;$$

and hence we may write

$$\mathcal{E}(p_1, p_2, p_3, p_4) = \begin{vmatrix} 0 & a^2 & (a+b)^2 & b^2 \\ a^2 & 0 & b^2 & (a+b)^2 \\ (a+b)^2 & b^2 & 0 & a^2 \\ b^2 & (a+b)^2 & a^2 & 0 \end{vmatrix}.$$

To show this determinant to be different from zero, we consider the more general symmetric determinant

$$\Delta = \begin{vmatrix} 0 & \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^2 & 0 & \delta^2 & \epsilon^2 \\ \beta^2 & \delta^2 & 0 & \zeta^2 \\ \gamma^2 & \epsilon^2 & \zeta^2 & 0 \end{vmatrix}.$$

If each row of this determinant be multiplied by the product of those three of the six elements not contained in the row, and each column of the determinant be divided by the product of the three elements contained in the column, it is found that

---

* Thus:

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & (12)^2 & (13)^2 \\ 1 & (21)^2 & 0 & (23)^2 \\ 1 & (31)^2 & (32)^2 & 0 \end{vmatrix} = [(12)+(23)+(31)][(12)+(23)-(31)] \cdot [(12)-(23)+(31)][(12)-(23)-(31)].$$

† A point p_2 is said to lie between two points p_1, p_3 if $p_1p_2+p_2p_3=p_1p_3$.

‡ The proof of the existence of such sets of four points, and their characterization is due to Karl Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), p. 125.

§ Menger, loc. cit., p. 127; see also Mathematische Zeitschrift, vol. 33, p. 408.

$$\Delta = \begin{vmatrix} 0 & \alpha\zeta & \beta\epsilon & \gamma\delta \\ \alpha\zeta & 0 & \gamma\delta & \beta\epsilon \\ \beta\epsilon & \gamma\delta & 0 & \alpha\zeta \\ \gamma\delta & \beta\epsilon & \alpha\zeta & 0 \end{vmatrix},$$

from which Δ is readily expressible in the factored form as

$$\Delta = (\alpha\zeta + \beta\epsilon + \gamma\delta)(\alpha\zeta + \beta\epsilon - \gamma\delta)(\alpha\zeta - \beta\epsilon + \gamma\delta)(\alpha\zeta - \beta\epsilon - \gamma\delta).$$

Turning now to the determinant $\mathcal{E}(p_1, p_2, p_3, p_4)$ and substituting $\alpha = \zeta = a$; $\beta = \epsilon = \alpha + b$; $\gamma = \delta = b$ in the expression for Δ, we find that

$$\mathcal{E}(p_1, p_2, p_3, p_4) = 16a^2b^2(a + b)^2(a^2 + ab + b^2)$$

and hence $\mathcal{E}$ is not zero.

CASE B. *Three of the four bordered minors are zero, and* $\mathcal{E}(p_1, p_2, p_3, p_4)$ *is zero.* We shall show that Case B is impossible.

From the hypothesis that three of the four bordered minors vanish, it follows that for three of the four triples $p_{i_1}, p_{i_2}, p_{i_3}$, one point lies between the two others. Then only one of the two following cases is possible.*

SUB-CASE α. *One point, let us say* p_1, *lies between each two of the three others.* Since $\mathcal{E}(p_1, p_2, p_3, p_4)$ vanishes, the four points do not form a pseudo-linear quadruple, and as D does not vanish, the four points are not linear. Hence the three points p_2, p_3, p_4 are not linear, and the following relations exist:

$$(12) = a, \quad (13) = b, \quad (14) = c,$$
$$(23) = a + b, \quad (24) = a + c, \quad (34) = b + c.$$

The determinant $D(p_1, p_2, p_3, p_4)$ now has the form

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & (a+b)^2 & (a+c)^2 \\ 1 & b^2 & (a+b)^2 & 0 & (b+c)^2 \\ 1 & c^2 & (a+c)^2 & (b+c)^2 & 0 \end{vmatrix};$$

* Menger, loc. cit., p. 107.

and substituting again in Δ

$$\alpha = a,\ \beta = b,\ \gamma = c,\ \delta = a + b,\ \epsilon = a + c,\ \zeta = b + c,$$

we find

$$\mathcal{E}(p_1, p_2, p_3, p_4) = -16a^2b^2c^2(ab+bc+ca) \neq 0,$$

which contradicts the hypothesis

$$\mathcal{E}(p_1, p_2, p_3, p_4) = 0.$$

Hence this sub-case is impossible.

SUB-CASE β. *The relations p_2 lies between p_1 and p_3; p_3 lies between p_2 and p_4; p_4 lies between p_3 and p_1 subsist; that is, we may write*

$$p_1p_2 + p_2p_3 = p_1p_3,$$
$$p_2p_3 + p_3p_4 = p_2p_4,$$
$$p_3p_4 + p_4p_1 = p_3p_1.$$

Then we have

$$(12) = a, \quad (13) = a + b, \quad (14) = a + b - c,$$
$$(23) = b, \quad (24) = b + c, \quad (34) = c;$$

and the determinant D takes the form

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & (a+b)^2 & (a+b-c)^2 \\ 1 & a^2 & 0 & b^2 & (b+c)^2 \\ 1 & (a+b)^2 & b^2 & 0 & c^2 \\ 1 & (a+b-c)^2 & (b+c)^2 & c^2 & 0 \end{vmatrix}.$$

From this, the determinant $\mathcal{E}(p_1, p_2, p_3, p_4)$ is found to have the value

$$\mathcal{E}(p_1, p_2, p_3, p_4) = 16b^2c^2(a + b)^2(ab + ac + b^2) \neq 0,$$

which again contradicts the hypothesis that $\mathcal{E}(p_1, p_2, p_3, p_4)$ vanishes. Thus this sub-case has been shown to be impossible, and Case B cannot exist. The theorem is then proved.

Since we have shown that for a determinant satisfying the conditions of the theorem the case B is impossible, it follows that Case A alone is valid. We have, then, the following corollaries:

COROLLARY 1. *A determinant satisfying the conditions of the theorem is necessarily of the form*

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & c^2 & b^2 \\ 1 & b^2 & c^2 & 0 & a^2 \\ 1 & c^2 & b^2 & a^2 & 0 \end{vmatrix},$$

where one of the three numbers a, b, c is the sum of the two others.

COROLLARY 2. *There is a double infinity of determinants that satisfy the conditions of the theorem.*

COROLLARY 3. *If a determinant satisfies the conditions of the theorem, then necessarily the unbordered fourth-order principal minor is different from zero.*

COROLLARY 4. *If D is a determinant satisfying the conditions of the theorem, and $\mathcal{E}$ is the unbordered fourth-order principal minor of D, then there exist three positive numbers a, b, c, one of which is the sum of the other two, such that*

$$D = -32a^2b^2c^2, \quad \mathcal{E} = -\tfrac{1}{2}(a^2 + ab + b^2)D.$$

In conclusion, the theorem with which this note is concerned may be formulated in the following manner:

If the symmetric determinant

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12} & r_{13} & r_{14} \\ 1 & r_{21} & 0 & r_{23} & r_{24} \\ 1 & r_{31} & r_{32} & 0 & r_{34} \\ 1 & r_{41} & r_{42} & r_{43} & 0 \end{vmatrix}, \qquad (r_{ij} = r_{ji} > 0,\ i \neq j),$$

has all five of the complementary minors of the five elements in the principal diagonal zero, then the determinant vanishes.

THE RICE INSTITUTE

MAPS OF CERTAIN CYCLIC INVOLUTIONS ON TWO-DIMENSIONAL CARRIERS

BY W. R. HUTCHERSON

1. *Introduction.* The following paper derives the fundamental properties of the involutions on an algebraic surface which have but a finite number of invariant points. Except for a few particular cases, they cannot be regarded as subcases of those having a curve of invariant points; they require one more equation for their definition, analogous to the singular correspondences on algebraic curves. They exist only on surfaces having particular moduli.

2. *Discussion of I_n.* Consider two surfaces $F(x)=0$ and $\Phi(x')=0$ with the property that any point P on $F(x)=0$ uniquely fixes a point P' on $\Phi(x')=0$ and, conversely, the point P' fixes n points $P_1 \equiv P, P_2, \cdots, P_n$ on F. There is thus set up an $(n, 1)$ correspondence between the points of $F=0$ and $\Phi=0$. Now any one of the n points $P_1, \cdots, P_n$ on $F=0$ definitely determines the whole group of n points to which it belongs. Hence, it will be said that F *contains an involution* I_n of order n, and that this I_n *belongs to* $\Phi(x')=0$.

There are two kinds of involutions; F may contain one or more curves, each point of which contains two or more coincidences of these n points $P_1, \cdots, P_n$, $P_i = P_k$. Such curves are called *curves of coincidences*. The surface $\Phi(x')=0$ then contains a locus of branch points in (1, 1) correspondence with the curve of coincidences on F. The other kind of involution is such that F has only a finite number of coincident points. Thus, $\Phi(x')$ has in this case exactly the same number of branch points.

If $\Phi(x')=0$ is a rational surface, or a plane, I_n is said to be *rational*. If $F(x)=0$ is rational, $\Phi(x')=0$ must be rational.* The converse is not true.

In this paper only I_n on $F(x)=0$ with a finite number of coincident points will be considered. Such an I_n can be gener-

* Castelnuovo, Mathematische Annalen, vol. 44 (1894), pp. 125–155.

ated by a group of birational transformations of the surface $F(x)=0$ into itself. The involution is cyclic, abelian, etc. according as it is generated by a cyclic, abelian, etc. group of transformations. This group is, of course, cyclic if n is prime. However, it may or may not be birational (Cremonian) for the whole space in which $F(x)=0$ lies.*

3. *Space of r Dimensions.* In a space S_r of r dimensions, it is possible to reduce $F(x)=0$ and $\Phi(x')=0$ to their normal forms. A surface is said to be *normal* in a linear S_r when it can not be obtained as the projection of a surface of the same order from a space S_m, $m>r$.

Call T the cyclic transformation which generates I_n (where n is a prime) on $F(x)=0$, and P a point of coincidence. Let C be any curve on $F(x)=0$, through P. The image of C under T is another curve on F through P. The point P is called a *perfect point of coincidence* if C and every image of C touches the same line at P for every tangent to F at P. Otherwise P is *non-perfect*. The corresponding branch point on $\Phi(x')=0$ is said to be perfect or non-perfect according as P is a perfect or non-perfect point of coincidence.

When F is reduced to its normal form, the operations of I_p can be represented by a collineation of period p, under which F is invariant in S_r. Since $p<r$, there exist not more than p spaces of invariant points. The form of each transformation is $x_i=\theta^{ri}x_i'$, where $\theta^p=1$.

4. *Discussion of I_2.* THEOREM 1. *If $n=2$, every point P of coincidence on $F(x)=0$ must be a perfect point.*

PROOF. It has been proved that, given an I_p, it can always be represented by a collineation in S_R, having p axes, or spaces of invariant points, only one of which meets the surface F which contains the involution.

When $p=2$ there are then only two axes. The transformation T may be written

$$X_1':X_2':\cdots:X_{r+1}':X_{r+2}':\cdots:X_{r+s+2}' = X_1:X_2:\cdots:X_{r+1}:\epsilon X_{r+2}:\cdots:\epsilon X_{r+s+2},$$

where $\epsilon=-1$.

* F. Enriques, Bologna Rendiconti, (2), vol. 14 (1910), pp. 71–75.

Consider an invariant point A on a surface F in S_R, where $R=r+s+1$. Call the hyperplanes

$$\sum_{i=1}^{r+1} s_i x_i' = 0,\ \Sigma^{(0)}, \text{ and } \sum_{k=r+2}^{r+s+2} r_k x_k' = 0,\ \Sigma^{(1)}.$$

Now $\Sigma^{(0)}$ contains one axis $S^{(1)}$ and $\Sigma^{(1)}$ the other $S^{(0)}$. Require $\Sigma^{(0)}$ to pass through point A in axis $S^{(0)}$. It cuts F in a system of invariant curves $|C|$, on each of which point A is the only coincident point. The tangents to $|C|$ at point A on F all lie in one tangent plane. $\Sigma^{(0)}$ contains this plane. There are two invariant directions at A in this plane or all are invariant. Call two invariant directions a_1 and a_2; then a_1 must have another invariant point A_1 on it. The point A_1 must lie in the other axis $S^{(1)}$ for A is the only point of $S^{(0)}$ in the tangent plane. Likewise a_2 must have another invariant point A_2. It must also lie in $S^{(1)}$. A linear combination of the coordinates of A_1 and A_2 gives ∞ points on line A_1A_2, each of which is a coincident point (invariant point). Line A_1A_2 lies in axis $S^{(1)}$. Hence, by joining A and every point on A_1A_2, we obtain ∞ invariant directions. Therefore point A is a perfect point. This holds for a rational and an irrational surface.

Two illustrations of I_2 are given. The first one uses the quadratic transformation $x_i=1/x_i'$ in S_2 with four fixed points $(\pm 1, \pm 1, 1)$. Nets* of cubic curves are mapped on the Cayley cubic surface with four nodes.

The second illustration maps a surface of Enriques of order six upon a double plane. The Cremona involution in S_3, $y_i=1/y_i'$, has eight fixed points $(1, \pm 1, \pm 1, \pm 1)$ on the surface F.

Several theorems follow, the proofs of which are omitted.†

THEOREM 2. *Consider on an algebraic surface F a point A, which is non-perfect in a cyclic involution of third order under*

* A. Emch, *On the invariant net of cubics in the Steinerian transformation*, this Bulletin, vol. 24 (1918), pp. 327–330. *On plane algebraic curves which are invariant under a quadratic Cremona transformation*, Tôhoku Mathematical Journal, vol. 21(1922), pp. 310–326.

† Earlier proofs of these theorems were also given by Godeaux. See Sisam, *Involutions of irrational surfaces*, Bulletin of the National Research Council, No. 63, vol. 14 (1928), pp. 295–309. Godeaux, Brussells Bulletin, 1927, pp. 524–543.

which F is invariant; it follows that in the domain of the first order of A, there are two distinct fixed points. These points are perfect coincidences of the involution.

THEOREM 3. *Consider a cyclic involution of prime order, having only a finite number of fixed points, belonging to an algebraic surface F, and possessing a non-perfect fixed point adjacent to which are two perfect fixed points; then this involution is of order three.*

A map of I_3 belonging to an irrational quintic surface upon an irrational surface in S_7, has been discussed. There are two perfect and five non-perfect coincidences.

5. *Discussion of I_5 belonging to F_3 in S_3.* Consider the surface

$$F_3(x_1x_2x_3x_4) \equiv ax_1^2x_3 + bx_2^2x_1 + cx_3^2x_4 + dx_4^2x_2 = 0,$$

in S_3, invariant under the cyclic collineation T of order five

$$x_1' : x_2' : x_3' : x_4' = x_1 : \epsilon^1 x_2 : \epsilon^2 x_3 : \epsilon^3 x_4, \qquad (\epsilon^5 = 1).$$

There are four invariant axes $S^{(0)}$, $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ each consisting of a point: $P_1 \equiv (1, 0, 0, 0)$, $P_2 \equiv (0, 1, 0, 0)$, $P_3 \equiv (0, 0, 1, 0)$, and $P_4 \equiv (0, 0, 0, 1)$. Each lies on the surface F, and since these are the only possible invariant axes, the surface F has only four points of coincidence.

Consider a curve C, not transformed into itself by T, and passing through P_1. Take a plane $x_3 + \lambda x_4 = 0$ of the pencil passing through P_1 and P_2, tangent to C. This plane $x_3 + \epsilon\lambda x_4 = 0$ by T and hence is non-invariant. The curve cut out on F by $x_3 + \lambda x_4 = 0$ is therefore non-invariant. The common tangent to the two curves is not transformed into itself. Hence, the two curves do not touch each other at P_1. Since C was a variable curve through P_1 satisfying the non-invariant property, it follows that P_1 is a non-perfect coincidence point. A similar argument shows that P_2, P_3, and P_4 are also non-perfect coincidence points. The following theorem is proved.

THEOREM 4. *The I_5 belonging to F_3 in S_3 has four non-perfect points of coincidence.*

Consider the complete system cut out on F by the quintic surfaces. Let $|A|$ be the system. Its dimension is 55, its genus is 31, and the number of variable intersections of two members of the system is 75. A curve A of this system is not in general transformed into itself by T. There are, however, five partial systems

in $|A|$ which are transformed into themselves. Call these $|A_1|$, $|A_2|$, $|A_3|$, $|A_4|$, and $|A_5|$. By use of $|A_1|$, we find

$$a_{11111}x_1^5 + a_{22222}x_2^5 + a_{33333}x_3^5 + a_{44444}x_4^5 + a_{12444}x_1x_2x_4^3 + a_{12223}x_1x_2^3x_3 + a_{13344}x_1x_3^2x_4^2 + a_{11233}x_1^2x_2x_3^2 + a_{11224}x_1^2x_2^2x_4 + a_{11134}x_1^3x_3x_4 + a_{23334}x_2x_3^3x_4 + a_{22344}x_2^2x_3x_4^2 = 0.$$

We refer the curves A_1 projectively to the hyperplanes of a linear space of eleven dimensions. We obtain a surface Φ, of order 15, with hyperplane sections of genus 7, as the image of I_5.

The equations of the transformation for mapping I_5 upon Φ in S_{11} are

$$\rho X_1 = x_1^5, \quad \rho X_4 = x_4^5, \quad \rho X_7 = x_1x_3^2x_4^2, \quad \rho X_{10} = x_1^3x_3x_4,$$
$$\rho X_2 = x_2^5, \quad \rho X_5 = x_1x_2x_4^3, \quad \rho X_8 = x_1^2x_2x_3^2, \quad \rho X_{11} = x_2x_3^3x_4,$$
$$\rho X_3 = x_3^5, \quad \rho X_6 = x_1x_2^3x_3, \quad \rho X_9 = x_1^2x_2^2x_4, \quad \rho X_{12} = x_2^2x_3x_4^2.$$

By eliminating ρ, x_1, x_2, x_3, x_4 from these twelve equations and from $F_3(x_1x_2x_3x_4)=0$, we get as the nine equations defining the surface Φ,

$$\begin{Vmatrix} X_4 & X_5 & X_{12} \\ X_5 & X_9 & X_6 \end{Vmatrix} = 0, \quad \begin{Vmatrix} X_2 & X_9 & X_{12} \\ X_6 & X_{10} & X_7 \end{Vmatrix} = 0, \quad \begin{Vmatrix} X_1 & X_{10} & X_8 \\ X_{10} & X_7 & X_{11} \end{Vmatrix} = 0,$$
$$\begin{Vmatrix} X_3 & X_{11} & X_7 \\ X_{11} & X_{12} & X_5 \end{Vmatrix} = 0,$$

and $aX_8+bX_6+cX_{11}+dX_{12}=0$. Designate by P_1' the branch point of Φ corresponding to the point P_1 on F. The coordinates of P_1' are all zero except X_1.

The curves A_1 on F pass through P_1 if $a_{11111}=0$. The tangent plane at P_1 to F is $x_2=0$. Now, the system of quintic surfaces passing through P_1 cuts $x_3=0$ in the curves $x_3=0$, $a_{22222}x_2^5 + a_{44444}x_4^5 + a_{12444}x_1x_2x_4^3 + a_{11224}x_1^2x_2^2x_4 = 0$. For general values of the constants this is a quintic curve with a triple point at P_1, two branches being tangent to the line $x_2=x_3=0$ and one to the line $x_3=x_4=0$. When $a_{12444}=a_{11224}=0$, the plane quintic curve breaks up into five lines through P_1. These are all distinct except when either $a_{22222}=0$ or $a_{44444}=0$, when they coincide with $x_3=x_4=0$ or $x_2=x_3=0$, respectively. Since P_1 is non-

perfect, the $|A_1|$ through P_1 must have five distinct branches, unless each branch touches one of the two invariant directions.

In the plane $x_3=0$ the involution I_5 is generated by the homography T_1, which is $x_1' : x_2' : x_4' = x_1 : \epsilon x_2 : \epsilon^3 x_4$. By use of the plane quadratic transformation S, which is $x_1 : x_2 : x_4 = z_1^2 : z_1 z_2 : z_2 z_4$ and its inverse $z_1 : z_2 : z_4 = x_1 x_2 : x_2^2 : x_1 x_4$, we can investigate the character of the adjacent invariant points along the two invariant directions at P_1. By the application of $ST_1S^{-1} \equiv T_1'$,

$$(z_1, z_2, z_4) \overset{S^{-1}}{\sim} (x_1 x_2,\ x_2,\ x_1 x_4) \overset{T_1}{\sim} (\epsilon x_1 x_2,\ \epsilon^2 x_2^2,\ \epsilon^3 x_1 x_4),$$

or

$$(x_1 x_2,\ \epsilon x_2^2,\ \epsilon^2 x_1 x_4) \overset{S}{\sim} (z_1,\ \epsilon z_2,\ \epsilon^2 z_4).$$

Thus the new transformation T_1' is $x_1' : x_2' : x_4' = x_1 : \epsilon x_2 : \epsilon^2 x_4$. The invariant point adjacent to P_1 along the line $x_3 = x_4 = 0$ is still a non-perfect coincidence point. Investigate the next point by use of $ST_1' S^{-1} \equiv T_1''$ $(z_1, z_2, z_4) \sim (z_1, \epsilon z_2, \epsilon z_4)$. This point is a perfect point of coincidence.

By use of another quadratic transformation R, namely $x_1 : x_2 : x_4 = z_1^2 : z_2 z_4 : z_1 z_4$ and its inverse $z_1 : z_2 : z_4 = x_1 x_4 : x_1 x_2 : x_4^2$, the adjacent point to P_1 along $x_2 = x_3 = 0$ can be investigated. Applying RT_1R^{-1} as above, we have $(z_1, z_2, z_4) \sim (\epsilon^2 z_1, z_2, z_4)$. Hence we get a perfect coincidence point. The following theorem is proved.

THEOREM 5. *The non-perfect coincidence point P_1 on F has one adjacent perfect point along the line $x_2 = x_3 = 0$, a non-perfect one along the line $x_3 = x_4 = 0$, with a perfect one adjacent to this.*

The tangent plane to F at $P_2(0, 1, 0, 0)$ is $x_1 = 0$. The homography T_2 in $x_1 = 0$ is $x_2' : x_3' : x_4' = x_2 : \epsilon x_3 : \epsilon^2 x_4$. Apply ST_2S^{-1} and proceed as above. We find $x_2' : x_3' : x_4' = x_2 : \epsilon x_3 : \epsilon x_4$. Hence, the adjacent point along $x_1 = x_3 = 0$ is perfect. By use of RT_2R^{-1} we obtain $(z_2, z_3, z_4) \sim (z_2, z_3, \epsilon_3 z_4)$. This indicates a non-perfect point. By use of $RT_2'R^{-1}$, we get $x_2' : x_3' : x_4' = \epsilon^3 x_2 : x_3 : x_4$. This gives a perfect point. Hence we may state the following theorem.

THEOREM 6. *The non-perfect coincidence point P_2 on F has one adjacent point along the line $x_1 = x_4 = 0$, a non-perfect adjacent one along $x_1 = x_3 = 0$, with a perfect one adjacent to this.*

The point $P_3(0, 0, 1, 0)$ has $x_4 = 0$ for its tangent plane. The homography becomes, in this tangent plane, T_3: $x_1' : x_2' : x_3'$

$=x_1:\epsilon x_2:\epsilon^2 x_3$. Introduce two quadratic transformations for use in discovering the nature of $P_3(0, 0, 1, 0)$. Calling the first one U and the second V, we have

$$\begin{aligned} U: &\quad y_1:y_2:y_3 = w_2w_3:w_1w_2:w_3^2, \\ U^{-1}: &\quad w_1:w_2:w_3 = y_2y_3:y_1^2:y_1y_3, \\ V: &\quad y_1:y_2:y_3 = w_1w_2:w_1w_3:w_2^2, \\ V^{-1}: &\quad w_1:w_2:w_3 = y_1^2:y_1y_3:y_2y_3. \end{aligned}$$

The adjacent points along the line $x_4=x_2=0$ compel the use of $UT_3U^{-1}\equiv T_3'$, then $VT_3'V^{-1}$ or T_3'', and then $UT_3''U^{-1}$ before a perfect point is found. We have $(w_1, w_2, w_3)\sim(\epsilon_2w_1, w_2, \epsilon^2w_3)$. This point is non-perfect. Consider $(w_1, w_2, w_3)\sim(\epsilon^3w_1, w_2, \epsilon^2w_3)$. This also is non-perfect.

Consider $(w_1, w_2, w_3)\sim(\epsilon^2w_1, \epsilon^2w_2, \epsilon^2w_3)$. This is a perfect point in the neighborhood of P_3 of the third order.

Now consider the possibilities along $x_4=x_1=0$, an invariant direction. Consider $(w_1, w_2, w_3)\sim(w_1, \epsilon^2w_2, \epsilon^3w_3)$. This is a non-perfect point. We have $(w_1, w_2, w_3)\sim(\epsilon^2w_1, \epsilon^2w_2, \epsilon^3w_3)$. Hence we may state the following theorem.

THEOREM 7. *The non-perfect coincidence point P_3 on F has no adjacent perfect point of coincidence. There is one perfect point in the domain of the second order of P_3 and another in the domain of the third order of P_3.*

The point $P_4(0, 0, 0, 1)$ has the plane $x_2=0$ for its tangent plane. The homography T in this plane becomes T_4: $x_1':x_3':x_4' = x_1:\epsilon^2x_3:\epsilon^3x_4$. Consider the direction $x_3=x_2=0$ at P_4. We have $(w_1, w_3, w_4)\sim(w_1, w_3, \epsilon^3w_4)$. Hence, the adjacent point is perfect along $x_2=x_3=0$. Now $(w_1, w_3, w_4)\sim(w_1, \epsilon^3w_3, w_4)$ and $(w_1, w_3, w_4)\sim(w_1, w_3, \epsilon^3w_4)$. Hence we have the following theorems.

THEOREM 8. *The non-perfect coincidence point P_4 has an adjacent perfect point along the line $x_2=x_3=0$, a non-perfect one along the line $x_1=x_2=0$, with a perfect one adjacent to this.*

THEOREM 9. *The system of invariant curves cut upon F by surfaces of degree lower than five all pass through the four coincidence points along the invariant directions. The number of branches through each point is less than five.*

CORNELL UNIVERSITY

THE DISTRIBUTIVE LAWS FOR HOMOGENEOUS LINEAR SYSTEMS

BY A. A. BENNETT

The terminology of addition and of multiplication may be readily extended to apply to homogeneous linear systems or to their geometrical interpretations, linear projective spaces. The product ab of two linear spaces a and b, denotes the intersection or greatest linear space common to the two given spaces. It is thus the logical product of the spaces. The sum $a+b$ denotes the join or least linear space containing the given spaces as subspaces. It is analogous to the logical sum but is distinguished from it by the fact that $a+b$ will in general contain also points which are in neither a nor b. Addition is commutative as is also multiplication. One has $a+a=a$, and $aa=a$, etc. Many of the formulas established for the algebra of logic hold here also.*

An essential feature of the algebra of logic is the following pair of relations, the first of which is applicable to ordinary algebra:

$$\text{I} \qquad a(b+c) = ab + ac,$$

$$\text{II} \qquad a + bc = (a+b)(a+c).$$

If I holds a is said to be distributive with respect to b and c in the first sense, if II holds, in the second sense. The proofs of these when based upon other postulates are not trivial.† For the case of even a one-dimensional projective space, these distributive relations fail to hold. Indeed if a, b, and c denote three distinct points of a line, and 0 denotes the null space, we have $b+c=$entire line, $a(b+c)=a$, but $ab=0$, $ac=0$, $ab+ac=0$. Hence $a(b+c)\neq ab+ac$. Similarly $a+bc=a$, but $(a+b)\ (a+c)$ $=$entire line.

While from two elements a and b, one can generate by addition and multiplication only the closed set ab, a, b, $a+b$, yet

* See A. A. Bennett, *Semi-serial order*, American Mathematical Monthly, vol. 37 (1930), pp. 418–423.

† See E. V. Huntington, *Postulates for the algebra of logic*, Transactions of this Society, vol. 5 (1904), pp. 288–309, particularly Peirce's proof, pp. 300–302.

from three elements, addition and multiplication yield in general an unlimited system. A special "distributive" system of 17 elements closed under addition and multiplication due to restrictions imposed upon a, b, c, is the following:

$$abc,\ ab,\ ac,\ bc,\ a(b+c),\ b(a+c),\ c(a+b),\ a,\ b,\ c,$$
$$a+bc,\ b+ac,\ c+ab,\ a+b,\ a+c,\ b+c,\ a+b+c,$$

where each of the three elements a, b, and c is distributive with respect to the other two in both senses, which as will be seen amounts to but one condition in ordinary spaces. The fact that it is actually closed under both operations may be verified by routine computation. The system is capable of realization by linear spaces in an unlimited number of ways. The following inequalities are readily proved from first principles:

$$\text{I}' \qquad a(b+c) \geqq ab+ac,$$
$$\text{II}' \qquad a+bc \leqq (a+b)(a+c),$$

but such inequalities do not tell the whole story.

In the formal algebra of logic there is no need of assuming the existence of prime non-zero quantities of such a sort that if p is a prime element (hence, $\neq 0$) and if $x<p$, then necessarily $x=0$. The existence of such prime elements, which serve as points in the geometrical interpretation, makes possible in the theory of spaces of at most a countable number of dimensions, a method of proof by mathematical induction, quite foreign in nature to the purely formal relations in the algebra of logic. In practice the theory of linear spaces of more than two dimensions is usually built up by use of mathematical induction from the theorems concerning collinear and coplanar points by use of transversality properties.

The following two theorems might appear to have a formal character independent of the existence of points. They are found to hold true for subspaces of any space of a countable number of dimensions, but no direct formal proof without the intervention of postulates comparable to that of the existence of points seems to be available.

THEOREM 1. *If three linear spaces a, b, and c are such that a is distributive with respect to b and c in one sense, it is distributive in both senses.*

THEOREM 2. *If three linear spaces a, b, and c are such that a is distributive with respect to b and c, then b is distributive with respect to a and c.*

These theorems will be proved only for spaces which contain points, although without the use of mathematical induction. The following property of points will be used.

PROPERTY. *If a, b, and c are non-empty spaces and x is a point of a that is in neither b nor c, but is in b+c, then there is a point y of b, and a point z of c, such that x is in y+z.*

Duality will not be available for if the total space be of infinitely many dimensions, it is not necessary to assume that the dual of a point has a meaning, although this is usually the case.

Two cases of the relations among a, b, c, will be considered.

CASE A. There exist three points x, y, z, such that simultaneously

(1) $ax = x$, (2) $x(ab + ac) = 0$, (3) $by = y$,

(4) $y(ab + bc) = 0$, (5) $cz = z$, (6) $z(ac + bc) = 0$

(7) $xy = xz = yz = 0$, (8) $x + y = x + z = y + z$.

CASE B. No set of points x, y, and z exists, such that simultaneously all eight conditions are satisfied.

In Case A, consider the relation between $a(b+c)$ and $ab+ac$. Since $ax=x$, $by=y$, $cz=z$, and $y+z=x+y$, it follows that x is in a, and in $b+c$, but from $x(ab+ac)=0$ it follows that x is not in $ab+ac$. Hence $a(b+c)>ab+ac$.

On the other hand, if $a(b+c)=ab+ac$, there cannot exist so much as a single point x in both a and $b+c$, but not in $ab+ac$. It remains to show that if $a(b+c)\neq ab+ac$, then necessarily all the conditions of Case A are satisfied. Since $a(b+c)\geqq ab+ac$, the hypothesis implies that $a(b+c)>ab+ac$, and there exists at least a point x in both a and $b+c$, while not also in $ab+ac$. Since x is in a but not in $ab+ac$, it is not in ab, and hence not in b. Similarly x is not in c. Hence there is in b a point y, and in c a point z such that x is in $y+z$. Hence we see that $ax=x$, $x(ab+ac)=0$, $by=y$, $cz=z$. It remains to prove the other relations required for Case A. For (7), we must show that x, y, z are distinct points. If $x=y$, then this common point is in both

a and b; hence in ab and hence in $ab+ac$, contrary to hypothesis. Similarly $x \neq z$. If $y=z$, then $y+z=y$. Since x is in $y+z$, x would coincide with y contrary to the previous conclusion. Hence (7) is established. Since x, y, z are distinct points, and x is in $y+z$, condition (8) follows. We must show that (4) and (6) are satisfied. We may first show that y is not in ab. For if it were, then the line $y+z$ which is the line $x+y$ would join two distinct points x, y, both in a, and hence lie in a, so that z would be in a, and hence in ac. Then $y+z$ would lie in $ab+ac$, and $x(ab+ac)$ would be x contrary to hypothesis. Similarly z is not in ac. If now y were in $ab+bc$ then y would be a fortiori in $ab+c$, as would also z, so that x would be in $ab+c$. But z is not in ab, since it is not in $ab+ac$, and x, which is in a, is not also in c since it is not in ac. Hence there would exist points y' in ab and z' in c, such that x is in $y'+z'$. But now as by the previous argument the line $y'+z'$ is the line $x+y'$ which must lie in a, so that z' is in ac, and x is then in $ab+ac$ contrary to hypothesis. Hence, $y(ab+ac)=0$. Similarly $z(ac+bc)=0$.

Hence Case B is a necessary and sufficient condition that a be distributive with b and c in the first sense.

Consider next distribution in the second sense. If a fails to be distributive with respect to b and c in the second sense, there is a point t in both $a+b$ and $a+c$, which is not in $a+bc$. This point cannot then be in a or in bc. It is therefore not in both b and c. If it is not in b, then we may choose a point x in a and a point y in b, such that t is in $x+y$, while t, x, y are distinct. If it is also not in c, then using the same x we can take x in c such that t is in $x+z$. If t is in c, we may take t as z. In any case we have three distinct collinear points x, y, z, in a, b, c, respectively. Hence (1), (3), (5), (7), (8) of Case A are satisfied. The line $x+y=x+z$ is not in $a+bc$, hence neither y nor z is in bc. Since t is not in both b and c, we may so assign the notation that it is not in b. Then x is not in ab, since otherwise $x+y$ would be in b, and t would also be in b, contrary to hypothesis. If now x were in ac, $x+z$ would be in c, so that y which is in this line would be in c, and hence in bc. Then $x+y$ would be in $a+bc$ contrary to hypothesis. Hence x is in neither ab nor ac. If x were yet in $ab+ac$, then the line $x+y=x+z$ would be in $ab+ac+b=ac+b$, and in $ab+c$. But the line could not lie wholly in ab nor ac nor b nor c, since x is in neither b nor c. The line must then join a

point of ac with a point of b and join a point of ab with a point of c. Let x_1 in ab, x_2 in ac, be the points where this line meets these spaces. The line then contains x, x_1, x_2 all in a, where although x_1 and x_2 might coincide, neither could be x. But then the line would lie wholly in a, contrary to hypothesis. Hence $x(ab+ac)=0$, and (2) is established. Likewise y cannot be in $ab+bc$. For if it were then $x+y$ would be in $a+ab+bc=a+bc$, contrary to hypothesis. Thus (4) is established, and (6) follows similarly. Hence for a to fail to be distributive in the second sense implies Case A. Conversely given Case A, then a fails to be distributive with respect to b and c in the second sense. Indeed y will then be in $a+b$ and also in $x+y$ which is in $a+c$. But y will not be in $a+bc$. For y is in b, but not in $ab+bc$, hence not in ab nor bc, hence not in a nor bc. If y were yet in $a+bc$, there would be a point u in a, and a point v in bc such that y would be in $u+v$. But y and v are then distinct and are both in b. Hence $u+v$ is in b, and u is in b. Hence u is in ab. Hence y would be in $ab+bc$ contrary to hypothesis. Hence Case A is a necessary and sufficient condition that a fail to be distributive with respect to b and c in the second sense.

Since Case B is necessary and sufficient for a to be distributive with respect to b and c in the first sense and again also in the second sense, Theorem 1 is proved. Since this Case B is symmetric in a, b, and c, Theorem 2 is proved.

BROWN UNIVERSITY

ON FACTORING LARGE NUMBERS*

BY D. H. LEHMER† AND R. E. POWERS

1. *Introduction.* Various non-tentative methods of factoring a given odd number N, based on the expansion of $N^{1/2}$ in a regular continued fraction, have been described.‡ The success of most of these methods depends on the appearance of a perfect square among the denominators of the complete quotients. In practice, however, such an event occurs all too infrequently. More often

* Presented to the Society, April 11, 1931.

† National Research Fellow.

‡ Dickson, *History of the Theory of Numbers*, vol. 1, Chapter 14; and D. N. Lehmer, this Bulletin, vol. 13, p. 501, and vol. 33, pp. 35–36.

it happens that the product of two or more denominators is a square. In this case two methods are available for obtaining a factorization of N. These methods are described and compared in the present paper.

Sometimes a denominator Q_n of the nth complete quotient has a prime factor in common with N. This factor appears also in the numerators of the nth and $(n+1)$st complete quotients, and is recognized by inspection. *We are therefore justified in assuming that as far as the expansion has been carried out, the numerators and denominators of the complete quotients are prime to N.* Under this assumption we prove that the two methods described below will both fail or both succeed in a given instance. We also show in §5 that a natural attempt to modify and combine the two methods is doomed to failure.

2. *The Method Using the P's.* In expanding the square root of N in a continued fraction we have for the general form of the nth complete quotient

$$x_n = (P_n + N^{1/2})/Q_n, \qquad (x_0 = N^{1/2},\ [x_n] = q_n).$$

From the familiar relation $P_n^2 + Q_nQ_{n-1} = N$, we have

$$(1) \qquad -Q_nQ_{n-1} \equiv P_n^2 \pmod{N}.$$

For $n=1$ this becomes $-Q_1 \equiv P_1^2$, since $Q_0 = 1$, and for $n=2$ we get

$$Q_2P_1^2 \equiv P_2^2 \pmod{N}.$$

In general, if we write $(-1)^nQ_n = Q_n^*$, we have

$$(2) \quad Q_n^*(P_{n-1}\cdot P_{n-3}\cdot P_{n-5}\cdots P_r)^2 \equiv (P_n\cdot P_{n-2}\cdot P_{n-4}\cdots P_s)^2 \pmod{N},$$

where $r=1$, $s=2$ or $r=2$, $s=1$, according as n is even or odd. To prove this, we assume that (2) is true for $n-1$, or

$$(3) \quad Q_{n-1}^*(P_{n-2}\cdot P_{n-4}\cdot P_{n-6}\cdots P_s)^2 \equiv (P_{n-1}\cdot P_{n-3}\cdot P_{n-5}\cdots P_r)^2,$$

and then show it true for n as follows. Multiplying (1) by

$$(P_{n-1}\cdot P_{n-3}\cdots P_r)^2\cdot(P_{n-2}\cdot P_{n-4}\cdots P_s)^2,$$

and dividing by (3), we get (2). But we have shown above that (2) is true for $n=1$, 2; hence it is true in general.

Two Q^*'s are said to be *equivalent* if their product is a square, that is, Q_i^* is equivalent to Q_j^* if $x^2Q_i^* = y^2Q_j^*$. From this equation we have by substituting $n=i$ and $n=j$ in (2), and noting that i and j are of the same parity,

$$(4)\quad (xP_{i+1}\cdot P_{i+3}\cdots P_{j-1})^2 - (yP_{i+2}\cdot P_{i+4}\cdots P_j)^2 \equiv 0 \pmod N.$$

Unless N divides either $(xP_{i+1}\cdot P_{i+3}\cdots P_{j-1}) \pm y(P_{i+2}\cdot P_{i+4}\cdots P_j)$, we obtain a factor of N by finding the G.C.D. of N and one of these numbers. If the two equivalent Q^*'s are near each other in the series of denominators, the factors of N will be disclosed with a minimum of effort.

This method may be extended to the case in which the product of more than two Q^*'s is a square. This involves a straightforward application of (2) as we illustrate later, and the ease with which the method may be applied depends again on the relative position of the Q^*'s and on the parities of their subscripts. It is of course unnecessary to compute the actual products of the P's involved, since these products may be reduced modulo N.

3. *The Method Using the A's.* If A_n/B_n is the nth convergent to $N^{1/2}$, we have the well known relation

$$A_{n-1}^2 - NB_{n-1}^2 = (-1)^nQ_n = Q_n^*,$$

which can be written

$$(5)\qquad Q_n^* \equiv A_{n-1}^2 \pmod N.$$

Hence if Q_i^* and Q_j^* are equivalent, so that $x^2Q_i^* = y^2Q_j^*$, then

$$(xA_{i-1})^2 - (yA_{j-1})^2 \equiv 0 \pmod N.$$

Unless N divides either $xA_{i-1} \pm yA_{j-1}$ it is possible by the greatest common divisor process to obtain a factorization of N. In the same way more than two Q^*'s may be used. For example, if $x^2Q_i^*Q_j^* = y^2Q_k^*$, then

$$(xA_{i-1}A_{j-1})^2 - (yA_{k-1})^2 \equiv 0 \pmod N.$$

The A's are of course calculated by the familiar recurrence

$$(6)\qquad A_n = q_nA_{n-1} + A_{n-2}, \qquad (A_{-2} = 0, A_{-1} = 1),$$

and when necessary the A's may be reduced modulo N.

4. *Comparison of the Methods.* The two methods are best compared by a numerical example. Consider the case $N = 13290059$.† The elements for $N^{1/2}$ are as follows.

n	P_n	Q_n*	q_n	$A_n \pmod N$
0	0	1	3645	3645
1	3645	$-2 \cdot 2017$	1	3646
2	389	3257	1	7291
3	2868	$-5 \cdot 311$	4	32810
4	3352	1321	5	171341
5	3253	$-2 \cdot 5^2 \cdot 41$	3	546833
6	2897	2389	2	1265007
7	1881	$-2 \cdot 13 \cdot 157$	1	1811840
8	2201	2069	2	4888687
9	1937	$-2 \cdot 5 \cdot 461$	1	6700527
10	2673	$31 \cdot 43$	4	5110677
11	2659	$-2 \cdot 2333$	1	11811204
12	2007	$5 \cdot 397$	2	2152967
13	1963	$-2 \cdot 2377$	1	674112
14	2791	$13 \cdot 89$	5	5523527
15	2994	-3739	1	6197639
16	745	$2 \cdot 13 \cdot 131$	1	11721166
17	2661	-1823	3	1490960
18	2808	$5 \cdot 593$	2	1413027
19	3122	$-5 \cdot 239$	5	8556095
20	2853	$2 \cdot 5 \cdot 431$	1	9969122
21	1457	-2591	1	5235158
22	1134	$41 \cdot 113$	1	1914221
23	3499	$-2 \cdot 113$	31	11415773
24	3507	$5 \cdot 877$	1	39935
25	878	$-5 \cdot 571$	1	11455708
26	1977	$2 \cdot 31 \cdot 53$	1	11495643
27	1309	$-13 \cdot 271$	1	9661292
28	2214	2381	2	4238109
29	2548	$-5 \cdot 571$	2	4847451
30	3162	1153	5	1895246
31	2603	$-2 \cdot 5^2 \cdot 113$	1	6742697
32	3047	709	9	9419283
33	3334	-3067	2	12291204
34	2800	1777	3	6422718
35	2531	$-2 \cdot 13 \cdot 149$	1	5423863
36	1343	$5 \cdot 593$	1	11846581

† This number is a factor of 254903331620, which in turn is the 55th term of the sequence 276, 396, 696, $\cdots$, in which each term is the sum of the aliquot parts of its predecessor.

The easiest method of factoring N is to apply the P method to the equal denominators $Q_{25}^* = Q_{29}^* = -2855$. For this case (4) reduces to

$$(P_{26} \cdot P_{28})^2 - (P_{27} \cdot P_{29})^2 \equiv 0 \pmod{N}.$$

The G.C.D. of N and $P_{26} \cdot P_{28} - P_{27} \cdot P_{29}$ is found to be 3119, and

$$N = 13290059 = 3119 \cdot 4261.$$

Similarly it is found that $Q_{18}^* = Q_{36}^* = 2965$, and the factors of N can be obtained in the same way, but with more effort, since these Q^*'s are 18 terms apart and hence more P's are involved. Also $Q_{23}^* = -2 \cdot 113$ is equivalent to $Q_{31}^* = -2 \cdot 5^2 \cdot 113$, from which the factors of N may be found.

The case in which three Q^*'s are involved is illustrated by Q_5^*, Q_{22}^* and Q_{23}^*, where the difference $5 \cdot P_2 P_4 P_{23} - 113 \cdot P_1 P_3 P_5$ has the factor 3119 in common with N. In each of these instances the A method is applicable. In the last case, for instance, we can write

$$(5 \cdot A_{21} \cdot A_{22})^2 - (113 \cdot A_4)^2 \equiv 0 \pmod{N},$$

from which the factors of N follow. This example shows the advantages of the P method when two equivalent Q^*'s appear near each other. When this is not the case, however, the method using the A's is more expeditious since the calculations are simpler.

We now show that the ease of application is the only deciding factor in choosing one of the two methods. To do this we make use of the following lemma.

LEMMA. *If n is any integer, then*

$$P_n + (-1)^n A_{n-1} A_{n-2} \equiv 0 \pmod{N}.$$

PROOF. For $n = 0, 1, 2$ this is true, though trivial. Supposing the lemma is true for $n-1$, we show it true for n itself as follows:

We have by assumption

$$P_{n-1} + (-1)^{n-1} A_{n-2} A_{n-3} \equiv 0 \pmod{N}.$$

Adding and subtracting $Q_{n-1} q_{n-1}$ and using the recurrence

$$P_n = Q_{n-1} q_{n-1} - P_{n-1}, \tag{7}$$

we find

$$\begin{aligned} 0 &\equiv P_{n-1} - Q_{n-1}q_{n-1} + (-1)^{n-1}A_{n-2}A_{n-3} + Q_{n-1}q_{n-1} \\ &\equiv -P_n + (-1)^{n-1}A_{n-2}(A_{n-3} + A_{n-2}q_{n-1}) \\ &\equiv -P_n + (-1)^{n-1}A_{n-2}A_{n-1}. \end{aligned}$$

That is

$$P_n + (-1)^n A_{n-1}A_{n-2} \equiv 0 \pmod{N},$$

which is the lemma.

THEOREM 1. *The success of one method in a particular instance implies the success of the other.*

PROOF. For simplicity the proof is given for the case of two equivalent Q^*'s. It can be easily amplified to cover the general case. Let Q_i^* and Q_j^* be equivalent, so that

$$x^2Q_i^* = y^2Q_j^*.$$

Suppose now that the A method succeeds and that the P method fails. Then N will divide either

$$(8) \quad (xP_{i+1}\cdot P_{i+3}\cdot P_{i+5}\cdots P_{j-1}) \pm y(P_{i+2}\cdot P_{i+4}\cdot P_{i+6}\cdots P_j).$$

Substituting for each P its value in terms of the A's as given by the lemma we have, after simplifying, either

$$xA_{i-1} \pm yA_{j-1} \equiv 0 \pmod{N}.\dagger$$

This implies the failure of the A method contrary to hypothesis. Therefore the P method succeeds.

By reversing the argument we may show that the success of the P method implies the success of the A method. This is done without using the assumption of §1 that the P's are prime to N. Hence we may restate Theorem 1 in a more precise form as follows.

The only instance of the success of one method and the failure of the other is that in which the A method succeeds, the P method fails, and a factor of N appears among the P's and Q's.‡

---

† The $\pm$ signs do not necessarily correspond to those in (8).

‡ An example of this instance is offered by $N=611$, $Q_6=Q_5=17$. Of course both methods fail when N is a prime, but the failure of either method should not be taken as an indication of the primality of N.

5. *Simultaneous Use of Both Methods.* Since each method finds a square to which Q_n^* is congruent mod N, one is naturally tempted to use both methods with any fixed Q whatsoever to obtain at once a difference of squares divisible by N. Unfortunately we have the following result.

THEOREM 2. *The two different methods for obtaining squares congruent to a particular denominator Q_k^* will not give a factorization of N if used together.*

PROOF. By the P method we have

$$P_{k-1}^2 \cdot Q_k^* \equiv P_k^2 \cdot Q_{k-2}^*. \tag{9}$$

This follows by writing (2) for $n=k$ and $n=k-2$ and taking their ratio. By the A method we have for the same Q_k^*

$$Q_k^* \equiv A_{k-1}^2 \pmod{N}. \tag{10}$$

Eliminating the Q^*'s from (9) and (10), we have the desired difference of squares

$$P_k^2 A_{k-2}^2 - P_{k-1}^2 A_{k-1}^2 \equiv 0 \pmod{N}.$$

But we now show that N always divides the sum

$$P_k A_{k-2} + P_{k-1} A_{k-1}. \tag{11}$$

In fact this sum, in view of (5), (6) and (7), is congruent to

$$q_{k-1} A_{k-2} \{ P_{k-1} + (-1)^{k-1} \cdot A_{k-2} A_{k-3} \}.$$

By the lemma with $n=k-1$, the quantity in brackets is divisible by N, hence N divides the sum (11).

Factorization is therefore possible only if N has a factor in common with the difference

$$A_{k-1} P_{k-1} - A_{k-2} P_k.$$

Such a factor would be common to

$$A_{k-1} P_{k-1} - A_{k-2} P_k + (A_{k-1} P_{k-1} + A_{k-2} P_k) = 2 A_{k-1} P_{k-1}.$$

But N is odd and prime to the P's and Q's, and, from (5), to the A's, hence the theorem.

STANFORD UNIVERSITY AND
DENVER COLORADO

DIVISION ALGEBRAS OVER AN ALGEBRAIC FIELD*

BY A. A. ALBERT

1. *Introduction.* H. Hasse has given a rigorous treatment of the theory of quadratic null forms over the field R of all rational numbers. He used the theory of p-adic numbers and readily extended his methods to obtain complete results on quadratic forms over any algebraic number field† and hence, by a simple isomorphism, over any field $R(\theta)$, where θ is any quantity satisfying an equation with coefficients in R and irreducible in R. Hasse has also used the fundamental principle of his quadratic form theory to prove several important theorems on cyclic (Dickson) algebras.‡

I have recently obtained theorems on rational division algebras by the use of A. Meyer's theorem that every indefinite quadratic form, with rational coefficients, in five or more variables is a null form.§ But now Hasse's theorems make the extension to algebras over algebraic fields $R(\theta)$ almost immediate. In particular it is shown here that the direct product of any two generalized quaternion algebras over $R(\theta)$ is never a division algebra,‖ and that a sufficient condition that a normal division algebra of order sixteen over $R(\theta)$ be a cyclic algebra is that it contain a quantity x not in $R(\theta)$ such that $x^2=\Delta_1^2+\Delta_2^2$ with Δ_1 and Δ_2 in $R(\theta)$.

---

* Presented to the Society, September 9, 1931.

† See his *Darstellbarkeit von Zahlen durch quadratische Formen in einem beliebigen algebraischen Zahlkörper*, Journal für Mathematik, vol. 153 (1923), pp. 113–130. These theorems seem to be not so well known in America as they should be. References on quadratic null forms are always made to the clumsy nineteenth century treatment of A. Meyer and the partial reproductions of Bachmann instead of to the modern short rigorous p-adic discussion of Hasse. Moreover Hasse's extensions to algebraic fields are of extreme importance and they should be better known.

‡ *Theory of cyclic algebras over an algebraic number field*, which will appear in the January, 1932, issue of the Transactions of this Society.

§ For references see §3.

‖ This theorem has several applications. I shall use it in a paper to be offered for publication to the Transactions of this Society to prove that all normal division algebras of order sixteen over a field $R(\theta)$ are cyclic (Dickson) algebras.

Hasse's new theorems on cyclic algebras are also used here to obtain an alternative proof of the above theorem on generalized quaternion algebras. In fact I show here that a necessary and sufficient condition that a direct product of any two normal division algebras over a field $R(\theta)$ be a division algebra is that their orders be relatively prime.

2. *Preliminary Theorems.* Let F be any non-modular field. We shall assume the following known theorems.*

THEOREM 1. *The order of any normal simple algebra is the square of an integer n called the degree of the algebra.* (WEDDERBURN.)

THEOREM 2. *Let A be a simple algebra over F and let B be a normal simple sub-algebra of A. Then A is the direct product $B \times C$ of B and another simple sub-algebra C of A.* (WEDDERBURN.)

THEOREM 3. *Every simple algebra A over F is expressible in the form $A = M \times D$ where M is a total matric algebra and D is a division algebra in one and only one way in the sense of equivalence, and conversely.* (WEDDERBURN.)

In the following theorems we take D to be a normal division algebra of order m^2, degree m, over its centrum F.

THEOREM 4. *The grade s of any quantity y of D is a divisor of the degree m of D and there exists a quantity x in D of grade m with respect to F such that y is in $F(x)$.* (ALBERT.)

THEOREM 5. *Every root in D of the minimum equation of a quantity y in D is a transform $z^{-1}yz$ of y by a quantity z in D.* (ALBERT.)

THEOREM 6. *The only quantities of D commutative with x in D of grade m with respect to F are quantities of $F(x)$.* (DICKSON.)

DEFINITION. *An algebraic field Z over F is called a splitting field (Zerfällungskörper) of D if $D \times Z$ is a total matric algebra.*

* For proofs of the very well known Theorems 1, 3, 6, see L. E. Dickson's *Algebren und ihre Zahlentheorie.* For Wedderburn's Theorem 2, see the Proceedings of the Edinburgh Mathematical Society, vol. 25 (1906), pp. 1–3. For Theorem 4, see the author's Theorem 37 of his paper, Annals of Mathematics, vol. 30 (1929), pp. 583–590, and for Theorem 5, see the same journal, vol. 30 (1929), pp. 322–338, Theorem 12. Finally for proofs of Theorems 7, 8, 11, 12 see the author's paper, *On direct products,* Transactions of this Society, vol. 33 (1931), pp. 690–711.

THEOREM 7. *Let Z be a splitting field of order n over F of D of degree m over F. Then $n=rm$, and if M is a total matric algebra of degree r the normal simple algebra $M\times D$ of degree n over F contains a sub-field $F(z)$ isomorphic to Z.* (ALBERT.)

THEOREM 8. *Let Y be an algebraic field of order n over F. Then $A\times Y=M\times B$, where M is a total matric algebra of degree s, and B is a normal division algebra of degree t over Y such that $m=st$, $n=sr$, so that s divides n.* (ALBERT.)

Suppose that y in D has grade s with respect to the centrum F of D. By Theorem 4 we have $n=st$ and there exists a quantity x in D of grade m for F such that y is in $F(x)$. The algebraic field $F(x)$ is then a relative field of order t over $F(y)$. If ξ is a scalar root of the minimum equation of x for F then, as is very well known, $F(\xi)$ is a splitting field for D. If the expression for y in $F(x)$ is $y=f(x)$, the scalar $\eta=f(\xi)$ has grade s for F and $F(\xi)$ is a relative field $K(\xi)$ of order t over $K=F(\eta)$. Also η is a scalar root of the minimum equation of y for F. If $D'=D\times K$, then $D''=D'\times K(\xi)=D\times F(\xi)$ is a total matric algebra. But $D'=M\times B$ where M is a total algebra of degree α and B is a normal division algebra of degree β over its centrum K, such that

$$m = \alpha\beta, \quad s = \rho\alpha.$$

Also $D''=D'\times K(\xi)=M\times B\times K(\xi)$ is a total matric algebra so that, by Theorem 8 applied to B, the integer β divides t. But $st=m=\alpha\beta$ and α divides s, β divides t. Hence $\alpha=s$, $\beta=t$. If η_0 is any other scalar root of the minimum equation of y with respect to F, then $D\times F(\eta_0)$ is simply isomorphic with $D\times F(\eta)$ and has exactly the same properties. We have therefore proved the following theorem.

THEOREM 9. *Let y in D have grade s for F so that $m=st$. Then, if η is any scalar root of the minimum equation of y for F,*

$$D \times F(\eta) = M \times B,$$

where M is a total matric algebra of degree s and B is a normal division algebra of degree t over its centrum $F(\eta)$.

Let H be a total matric algebra equivalent to M of Theorem 9. As the author has shown (*On direct products*, loc. cit.) there ex-

ists a quantity η in H satisfying the condition of Theorem 9 so that $D\times H$ contains $D\times F(\eta)$ as a sub-algebra. But $D\times F(\eta) = M\times B$, so that, by Theorem 2, $H\times D = M\times C$, where C is necessarily a normal simple algebra. It follows from the uniqueness in Theorem 3 that C is equivalent to D. Algebra C is the algebra of all quantities of $H\times D$ which are commutative with all of the quantities of M, and contains B as a sub-algebra. If y_1 in D corresponds to η in C under the isomorphism between D and C, then y_1 is a root of the minimum equation of y with respect to F and, by Theorem 5, is a transform $z^{-1}yz$ of y by a quantity z in D. If the correspondence between the quantities of D and C is designated by $d \cong c$, then we can obviously set up a new simple isomorphism $zdz^{-1} \cong c$ in which now y corresponds to η. Hence D contains an algebra B_0 which is a normal division algebra of degree t over its centrum $F(y)$ equivalent to B as over $F(\eta)$. The algebra G of all quantities of D commutative with y has B_0 as a sub-algebra and, by Theorem 2, $G = B_0\times Q$, where Q is an algebra over $F(y)$. By Theorem 4 algebra B_0 contains a quantity b of grade t with respect to $F(y)$. The field $F(b, y)$ has order st over F and contains a quantity x of grade m with respect to F generating it. But x is in B_0 and is commutative with all of the quantities of Q. By Theorem 6 the quantities of Q are in $F(x)$, and hence in B_0, so that B_0 is G.

THEOREM 10. *The algebra B of Theorem 9 is simply isomorphic with the algebra of all quantities of D commutative with y, under a correspondence where y corresponds to η. Hence this latter algebra is a normal division algebra of degree t over its centrum $F(y)$.*

We shall also use the following theorem of Brauer which I discovered independently and which I have proved in a short new way in my paper *On direct products*, loc. cit.

THEOREM 11. *Write $m = p_1^{e_1}\cdot p_2^{e_2}\cdots p_t^{e_t}$, where the p_i are distinct primes. Then $D = D_1\times D_2\times \cdots \times D_t$, where D_i is a normal division algebra of degree $p_i^{e_i}$ over F in one and only one way in the sense of equivalence, and conversely.* (BRAUER, ALBERT.)

We require in addition the following theorem of my own.

THEOREM 12. *Let $m = p^e$, p a prime, and let x in D have grade m for F. Then there exists an algebraic field Z of order n over F, such that n is prime to p, $D\times Z$ is a normal division algebra of de-*

gree m over its centrum Z, and Z(x) is a cyclic field of order p over a sub-field Z(y) of order p^{e-1} over Z. (ALBERT.)

The following statement is an immediate consequence of Theorem 12, the definition of a cyclic algebra and Theorem 10.

THEOREM 13. *Let $D'=D\times Z$, y and x be as in Theorem* 12. *Then the algebra B of all quantities of D' commutative with y is a cyclic algebra of degree p over its centrum Z(y).*

3. *Applications of the Hasse Theory of Quadratic Forms.* Let R be the field of all rational numbers, and let $\phi(\xi)=0$ with coefficients in R and degree m be irreducible in R. Let

$$\xi_1, \cdots, \xi_r, \quad \eta_1, \cdots, \eta_s, \quad \bar{\eta}_1, \cdots, \bar{\eta}_s \tag{1}$$

be the $m=r+2s$ complex roots of the equation $\phi(\xi)=0$, where $\xi_1, \cdots, \xi_r$ are all real and $\eta_1, \cdots, \eta_s$ are all imaginary. Suppose that θ is any quantity such that $\phi(\theta)=0$ so that we can define an algebraic field $R(\theta)$ of order m over R, which is evidently simply isomorphic with each of the algebraic number fields generated by each of (1). An n-ary quadratic form

$$Q(\theta) \equiv Q(\theta; x_1, \cdots, x_n) \equiv \sum_{i,j}^{1,\cdots,n} a_{ij}(\theta)x_ix_j \tag{2}$$

with coefficients in $R(\theta)$ is called a null form if there exist $x_1, \cdots, x_n$ not all zero in $R(\theta)$ such that $Q=0$. Hasse has proved that *an n-ary quadratic form* (2), $n\geqq 5$, *is a null form if and only if the corresponding real forms*

$$Q_k \equiv Q(\xi_k) = \sum_{i,j}^{1,\cdots,n} a_{ij}(\xi_k)x_ix_j, \qquad (k = 1, \cdots, r),$$

are all indefinite.

We apply Hasse's theorem first to the form

$$Q = \alpha x_1^2 + \beta x_2^2 - \alpha\beta x_3^2 - (\rho x_4^2 + \sigma x_5^2 - \sigma\rho x_6^2). \tag{3}$$

For a fixed k, if we use the notation $\lambda(\xi_k)=\lambda_k$, then in Q_k the numbers $\alpha_k, \beta_k, -\alpha_k\beta_k$ all have the same sign only when all are negative. The numbers $-\rho_k, -\sigma_k, \sigma_k\rho_k$ are never all negative, so that the forms Q_k are all indefinite. Hence the form (3) is always a null form. I have proved* that if B and C are any two generalized quaternion algebras

* This Bulletin, vol. 37 (1931), pp. 301–312, in particular, p. 311.

$$B = (1, i, j, ij), \quad i^2 = \alpha, \quad j^2 = \beta, \quad ji = -ij,$$
$$C = (1, I, J, IJ), \quad I^2 = \rho, \quad J^2 = \sigma, \quad JI = -IJ,$$

then the direct product $B \times C$ is a division algebra if and only if the form (3) is not a null form. But when B and C are algebras over $R(\theta)$ we have shown that (3) is always a null form. Hence we have proved the following result.

THEOREM 14. *Let B and C be generalized quaternion algebras over a field $R(\theta)$. Then $B \times C$ is not a division algebra.*

In a recent paper* I proved that a sufficient condition that a normal division algebra D of order sixteen over R be a cyclic (Dickson) algebra, is that D contain a quantity x not in R but such that $x^2 = \Delta_1^2 + \Delta_2^2$ with Δ_1 and Δ_2 in R. The proof was rational throughout except for the use of the analog to Theorem 14 on page 184, which could have been avoided, and the essential proof that the form numbered (53) was a null form. The proof was exactly of the same nature as our proof that (3) is a null form, depending only on intrinsic *signs* of coefficients, and is valid for any field $R(\theta)$. We may state then the validity of the following theorem without further proof.

THEOREM 15. *A sufficient condition that a normal division algebra D of order sixteen over $R(\theta)$ be a cyclic (Dickson) algebra is that D contain a quantity x not in $R(\theta)$ such that $x^2 = \Delta_1^2 + \Delta_2^2$, where Δ_1 and Δ_2 are in $R(\theta)$.*

I have recently published two papers,† of which the results and the proofs given *are again valid for any field* $R(\theta)$, except that Theorem 4 of the latter paper, where the Meyer theorem on quaternary quadratic null forms is applied, should be replaced by the Hasse criteria for quaternary quadratic forms over an algebraic number field isomorphic with $R(\theta)$.

We shall now pass to a generalization and alternative proof of Theorem 14.

4. *The Direct Product $D \times B$.* Let A be any normal simple algebra over an algebraic field $F = R(\theta)$. By Theorem 3, $A = M \times D$,

* Transactions of this Society, vol. 32 (1930), pp. 171–195.

† *A necessary and sufficient condition for the non-equivalence of any two rational generalized quaternion division algebras*, this Bulletin, vol. 36 (1930), pp. 535–540, and *A construction of all non-commutative rational division algebras of order eight* Annals of Mathematics, vol. 31 (1930), pp. 567–576.

where M is a total matric algebra and D is a normal division algebra, the division algebra component of A. Hasse in his paper on cyclic algebras (loc. cit.) has called two normal simple algebras similar if they have the same division algebra components. Hasse has called a normal simple algebra cyclically representable if it is similar to a cyclic algebra and has proved the following theorems.

THEOREM 16. *Every cyclically representable algebra D over $R(\theta)$ is a cyclic algebra.* (HASSE.)

THEOREM 17. *The direct product of two cyclically representable algebras is cyclically representable.* (HASSE.)

We also use the following known theorem (*On direct products*, loc. cit.).

THEOREM 18. *Let D be any normal division algebra over a non-modular field. Then there exists an integer ρ which divides the degree m of D, such that ρ is the least integer α for which the direct product D^α is a total matric algebra. In particular D^m is a total matric algebra.* (BRAUER, ALBERT.)

The integer ρ is called the *exponent* of D. Hasse has also proved the following theorem.

THEOREM 19. *The exponent of any cyclic normal division algebra D of degree m over $F=R(\theta)$ is m.* (HASSE.)

Let D and B be any two cyclic normal division algebras of the same degree p, a prime, over their common centrum $F=R(\theta)$, an algebraic field. Then $D\times B$ is a normal simple algebra so that $D\times B=M\times C$, where M is a total matric algebra and C is a normal division algebra over F. Now $(D\times B)^p=D^p\times B^p$ is a total matric algebra by Theorem 18. Hence $(M\times C)^p$ is a total matric algebra so that C^p is a total matric algebra and the exponent of C is at most p. By Theorem 17 the algebra $M\times C$ and hence C is a cyclically representable algebra. By Theorem 16 algebra C is a cyclic algebra. By Theorem 19 the degree of C is its exponent so that the degree of C is at most p. But the degree of $M\times C$ is p^2 so that the degree of M is at least p and $D\times B$ is not a division algebra.

LEMMA. *The direct product of any two cyclic normal division algebras of the same degree p, a prime, is not a division algebra.*

Let next D and B be any normal division algebras of degrees p^e and p^f respectively, p a prime and both e and f not zero, over a common centrum $F=R(\theta)$. We may extend the centrum F of D and B to be Z, by Theorem 12, such that D' over Z has the property of Theorem 13, and this extension may be obviously made so that simultaneously B' has the property of Theorem 13. If $D\times B$ is a normal division algebra, then, since the order of Z with respect to F is prime to p, by Theorem 8, algebra $D'\times B'$ over Z is a normal division algebra. Let y in D and η in B be the quantities of Theorem 13. Then the algebra C of all quantities of D' commutative with y is a cyclic normal division algebra of degree p over its centrum $Z(y)$. The algebra E of all quantities of B commutative with η is a cyclic normal division algebra of degree p over its centrum $Z(\eta)$. The quantity y is in the normal division algebra $D'\times B'$ and the algebra of all quantities of $D'\times B'$ commutative with y is evidently $C\times B$ over its centrum $K=Z(y)$. In $C\times B$ over K the algebra of all quantities of this algebra commutative with η is evidently $C\times E$ over its centrum $K(\eta)$. But C and E are cyclic algebras of degree p over $K(\eta)$, an algebraic field over R and by our Lemma not a division algebra. This is a contradiction of the fact that every sub-algebra of a division algebra is a division algebra. Hence $D\times B$ is not a division algebra.

Now let D and B be any normal division algebras of degrees m and r respectively over $F=R(\theta)$. If m and r have a prime factor p in common, then $D=D_1\times D_2$, $B=B_1\times B_2$, where D_1 is a normal division algebra of degree p^e, the highest power of p dividing m, B_1 is a normal division algebra of degree p^f, the highest power of p dividing r. But $D\times B=(D_1\times B_1)\times(D_2\times B_2)$, while $D_1\times B_1$ is not a division algebra as we have proved. Hence $D\times B$ is not a division algebra. Conversely let the orders of D and B be relatively prime. By Theorem 11, algebra $D\times B$ is a normal division algebra. Passing from degrees to orders, we have the following result.

THEOREM 20. *A direct product of two normal division algebras over the same centrum $R(\theta)$ is a division algebra if and only if the orders of the two algebras are relatively prime.*

THE UNIVERSITY OF CHICAGO

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

COLUMBUS, OHIO, November 27-28, 1931.

Abstracts must be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 5. All abstracts received by that date will appear in the November issue of this Bulletin. By invitation of the program committee, Professor W. C. Graustein will speak on *Parallelism of Levi-Civita in classical differential geometry*, and Professor H. M. Gehman will speak on *Homeomorphic geometry*.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 28, 1931.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than October 31. All abstracts received by that date will appear in the November issue of this Bulletin. By invitation of the program committee, Professor R. L. Moore will deliver an address entitled *A set of Axioms for plane analysis situs*.

NEW ORLEANS, LOUISIANA, ANNUAL MEETING, December 28-31, 1931.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than November 28, 1931. Abstracts received by November 5 will appear in the November issue of the Bulletin. Professor E. R. Hedrick will deliver his retiring presidential address on the topic *Non-analytic functions of a complex variable*. There will be two joint sessions with the American Physical Society, and the ninth Josiah Willard Gibbs Lecture will be delivered at one of these by Professor P. W. Bridgman of Harvard University. At these joint sessions also, Professor G. A. Bliss will deliver his retiring vice-presidential address for Section A of the A.A.A.S., Professor W. F. G. Swann will read a paper by invitation, and Professor George D. Birkhoff will speak by invitation on the topic *Stability and instability of physical systems*. The Cole Prize in the Theory of Numbers will be awarded.

The usual February Meeting in New York City will be omitted.

NEW YORK CITY, March 25-26, 1932.

Abstracts are due by March 5, 1932.

CHICAGO, April 8-9, 1932.

Abstracts are due by March 18, 1932.

STANFORD UNIVERSITY, CALIFORNIA, April 9, 1932.

Abstracts are due by March 12, 1932.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

WHOLE NUMBER 385

CONTENTS

	PAGE
Analytic Functions and Mathematical Physics. By G. Y. Rainich	689
Application of Boolean Algebra to proving Consistency and Independence of Postulates. By B. A. Bernstein	715
A Non-Dense Plane Continuum. By J. H. Roberts	720
On a Theorem of von Sterneck. By D. H. Lehmer	723
A Note on Cyclic Algebras of Order Sixteen. By A. A. Albert	727
An Acknowledgment. By G. W. Starcher	730
On a Covariant Differentiation Process. By H. V. Craig	731
On the Divisibility of Locally Connected Spaces. By G. T. Whyburn	734
Quadratic Addition Theorems for Even Functions. By J. D. Grant	737
Transformations of Double Sequences with Application to Cesàro Summability of Double Series. By C. R. Adams	741
Useful Functions Associated with Rational Cubic Curves. By Clifford Bell	749
An Application of Metric Geometry to Determinants. By L. M. Blumenthal	752
Maps of Certain Cyclic Involutions on Two-Dimensional Carriers. By W. R. Hutcherson	759
The Distributive Laws for Homogeneous Linear Systems. By A. A. Bennett	766
On Factoring Large Numbers. By D. H. Lehmer and R. E. Powers	770
Division Algebras over an Algebraic Field. By A. A. Albert	777

For official Communications and Notices, see the inside of the back cover.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN